TEM with anomalous diffusion in fractal conductive media

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Abstract

The transient electromagnetic response of an inductive loop source over a half-space with fractal characteristics is simulated. The conductivity of the ground has a spatial distribution, which is described by a roughness parameter. The roughness can be related to the fractal dimension and controls the transient decay of the signal. Asymptotic limits are observed for the early and late time behaviour.

The work is based on a paper by Everett (2009), which has been reviewed and the aspect of asymptotic limits in time has been further investigated. Contrary to that paper, here a relationship between roughness and the electromagnetic decay at early and late time has been deduced. The transient decay for normal diffusion is $t^{-5/2}$ and for anomalous diffusion the decay is slower. The new power law for anomalous diffusion has been proven by theoretical analysis and has been verified by numerical experiments.

The numerical evaluation of the inverse Laplace transform with the method of the fast Hankel transform are excellent in numerical accuracy and the method with the Gaver-Stehfest algorithm is not sufficient to estimate the power law decay at late times.

Anomalous Diffusion

The concept of anomalous diffusion is a useful approach for the description of diffusion process and transport dynamics in complex systems. The fractional equations are derived asymptotically from basic random walk models and become a complementary tool for handling non-exponential relaxation patterns.

For transient electromagnetic diffusion Everett starts this concept with a generalized Ohm's law (Everett 2009, Weiss and Everett 2007)

$$j = \sigma E \rightarrow j = \sigma_\beta * E \equiv \int_0^t \frac{\sigma_\beta E(\tau)}{(t-\tau)^{1-\beta}} d\tau$$

the parameter $\sigma_\beta \sim \sigma t^{-\beta}$ describes the generalized electrical conductivity and is appropriate for the anomalous diffusion coefficient. The Ohm's law becomes a convolution between the generalized conductivity and the electric field E. The roughness parameter can vary $0 \leq \beta \leq 1$.

After applying Ampere‘s and Faraday‘s law to the generalized current density, we get the fractional diffusion equation

$$\nabla^2 E = \mu_0 \frac{\partial}{\partial t} \sigma_\beta * E = \mu_0 \sigma_\beta \ {}_0D_t^{1-\beta} \ E(t)$$

where now the fractional derivative or Riemann-Louiville operator is introduced (Metzler and Klafter 2000)

$${}_0D_t^{1-\beta} E(t) = \frac{1}{\Gamma(\beta)} \frac{\partial}{\partial t} \int_0^t \frac{E(\tau)}{(t-\tau)^{1-\beta}} d\tau$$

and Γ is the Gamma function which serves as normalization constant.
The fractional diffusion equation will be solved in the Laplace domain and the transformation of the operator yield to

$$L\left\{ {}_0D_t^{1-\beta} E(t) \right\} = s^{1-\beta} \widetilde{E}(s)$$

with the complex Laplace variable $s=i\omega$ (Abramowitz & Stegun 1964).
The fractional diffusion equation becomes now a simple expression

$$\nabla^2 \widetilde{E} = s^{1-\beta} \mu_0 \sigma_\beta \widetilde{E}(s)$$

This differential equation can be solved straightforward with the standard methods as used for the electromagnetic 1-D formulation in a layered medium.

Electromagnetic responese of a loop over rough half-space

Everett (2009) has used a separate horizontal loop configuration for transmitter and receiver as it is typically used for TEM. The transmitter loops usually a square loop can be represented by a circular loop whith equivalent area. For the separate loop configuration the time derivative of the vertical magnetic field is measured outside the transmitter loop.
In frequency domain the vertical magnetic field is presentated as Hankel Integrals

$$\widetilde{h}_z(s) = Ia \int_0^\infty \frac{\lambda^2}{\lambda + \sqrt{k_\beta^2 + \lambda^2}} J_1(\lambda a) J_0(\lambda r) d\lambda \ .$$

The integration is over the spatial wavenumber λ and I is the current in the transmiter, a the transmitter loop radius, r the distance to the receiver point, J_0 and J_1 are Bessel functions of order *0* and *1* and k_β the fractional wavenumber. The integral can be solved analytically for an infinitessimally magnetic dipole, therefore the limit of the first order Bessel function for small radius is taken into account (Abramowitz & Stegun 9.1.10)

$$J_1(\lambda a)\xrightarrow[a\to 0]{}\frac{\lambda a}{2}.$$

The response of a vertical magnetic dipole in frequency domain over a rough conductive media is

$$\widetilde{H}_z(s)=\frac{m}{2\pi\, r^3}\frac{1}{u^2}\left[-9+\left(9+9u+4u^2+u^3\right)e^{-u}\right]$$

with the fractional induction number $u = k_\beta r = (s^{1-\beta}\mu_0\sigma)^{1/2}\, r$.

Asymptotic limits in frequency and time domain

For normal diffusion with roughness equal zero the transient response can be also given in analytical expression but for a general fractional induction number with the roughness parameter β the transformation to time domain has to be done by numerical techniques. Only the asymptotic limits for the high and low frequency limit can be calculated in closed form. For high frequency and early time we get

$$\widetilde{H}_z^{h.f.}(s)=-\frac{m}{2\pi\, r^2}\frac{9}{u^2} \quad\leftrightarrow\quad H_z^{e.t.}(t)=-\frac{m}{2\pi\, r^3}\frac{9}{\mu_0\sigma\, r^2}\frac{1}{\Gamma(1-\beta)}\, t^{-\beta}.$$

The low frequency can be developed in a series

$$\widetilde{H}_z(s)=\frac{m}{2\pi\, r^3}\frac{1}{2}\left[-1-2\sum_{n=4}^{\infty}\frac{(n-1)(n-3)^2}{n!}\,(-u)^{n-2}\right].$$

The first 3 terms are important for the late time behavior

$$\widetilde{H}_z(s)=\frac{m}{2\pi\, r^3}\frac{1}{2}\left[-1-\frac{1}{2}u^2+\frac{4}{15}u^3\right]-+\ldots.$$

In time domain the first term is a δ-function without influence on late time decay. The second term is responsible for the late time behavior in a rough medium

$$L^{-1}\left(s^{1-\beta}\right)=\begin{cases}\delta'(t) & \text{for } \beta=0\\[2ex] -\dfrac{\Gamma(\beta)}{\beta-1}\, t^{-(2-\beta)} & \text{for } \beta>0\end{cases}$$

The third term describes the classical -5/2 decay response in a non-fractional medium and for a fractional medium this term decays faster than the previous second term.

The late time response for the magnetic field can be summarized

$$\dot{H}_z^{l.t}(t) = \frac{m}{2\pi r^3} \begin{cases} \dfrac{(\mu_0\sigma)^{3/2}}{10\sqrt{\pi}}\, t^{-5/2} & \text{for } \beta = 0 \\ \dfrac{\mu_0\sigma_\beta}{2}\dfrac{\Gamma(\beta)}{\beta-1}\, t^{-(2-\beta)} & \text{for } \beta > 0 \end{cases}$$

The cause for the heavy tailed decay response in rough geological media can be explained by the additive second term which yield to a continuous transition from non-fractional to fractional diffusion as is shown in figure 1 and 2.

Numerical inverse Laplace transform

Numerical methods are applied for the transformation to time domain. Since the transient is a real and causal function, the inverse Fourier or Laplace transform can be calculated by a sine-transform.

$$\dot{h}_z(t) = -\frac{2}{\pi}\int_0^\infty \mathrm{Im}\{\tilde{h}_z(\omega)\}\sin(\omega t)\,d\omega$$

In this expression I have already considered that the current step in the transmitter decribed by *1/iω* and the time derivative of the receiver coil by a multiplication with *iω* cancel each other. Since for diffusion processes the kernel function is a smooth function, the technique of the Fast Hankel Transform can be applied. The sine function is experessed by Bessel function with fractional order ½

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin(x).$$

The fast Hankel transform is a well know technique for calculating the transient response, I used 250 filter coefficients, 15 /decade, calculated with the program by Christensen (1990).

Everett (2009) has chosen the Gaver-Stehfest algorthim for his investigation. This method is also good for diffusion processes and has been succesfully applied. It is a favourized method in hydrology, because it needs only real arithmetic and the Laplace variable *s* is considered as a real variable. Knight and Raiche (1982) introduced this technique to the electromagnetic community. Stehfest (1970) published an Algol routine, which can be straight forward translated to other computer languages and Everett (2009) has tabled the coefficients for several total number of coefficients.

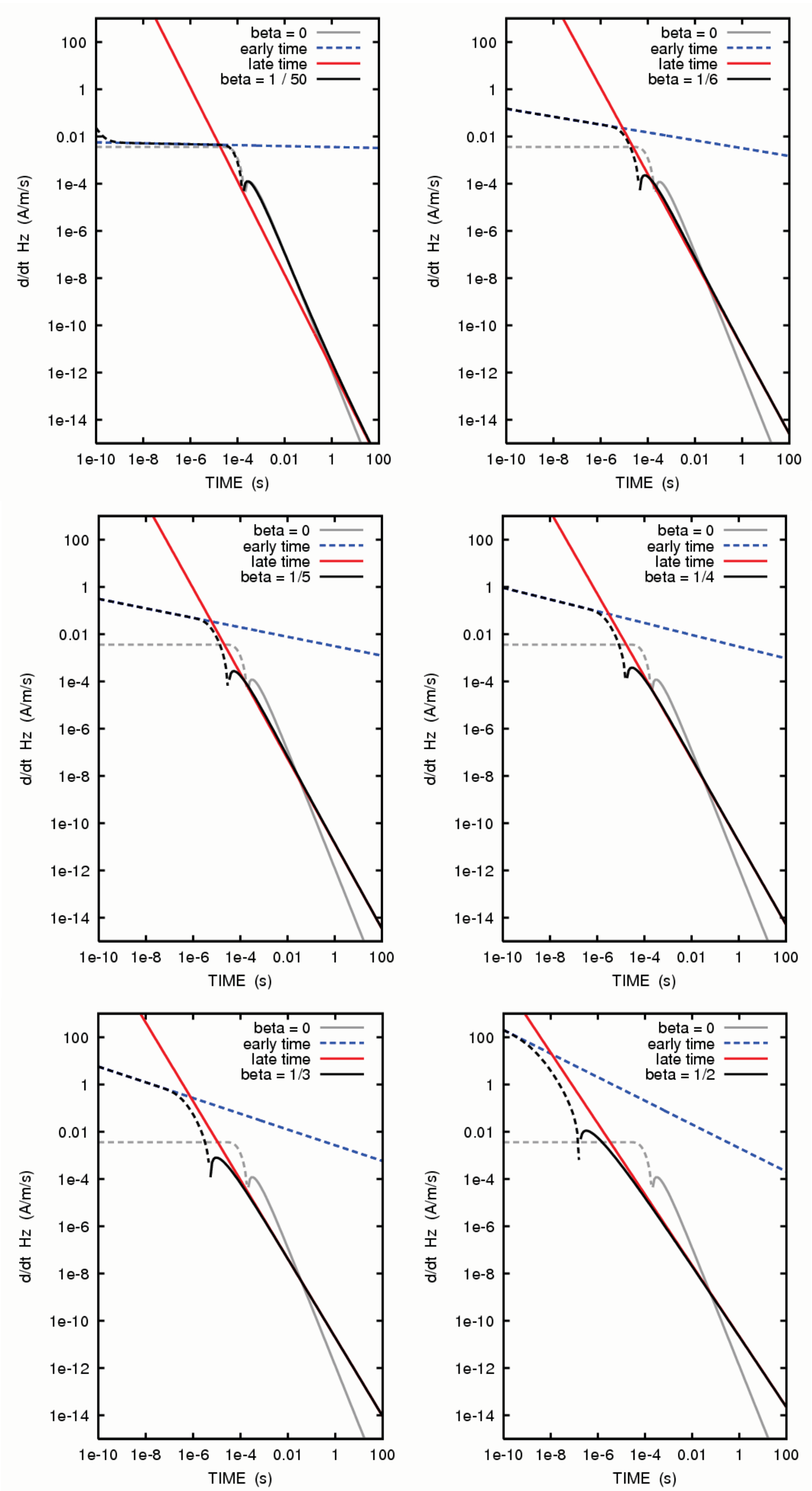

Figure 1: Transient loop-loop response with early and late time approximation over half-space with different roughness in conductivity, the offset is *100 m*, *σ = 0.1 S/m*.

The disadvantage of the Gaver-Stehfest algorithm is that the numerical accurracy cannot generally be increased by increasing the number of coeffcients. The accuracy is depending on the accurracy in number of digits of the kernel function and the number of digits of the maschine precision. Stehfest recommended 8 coefficients for single precision and 18 coeffcients for double precision. Everett used 18 coefficients. In my experiments I have found out that 12 is an optimal number for electromagnetic application.

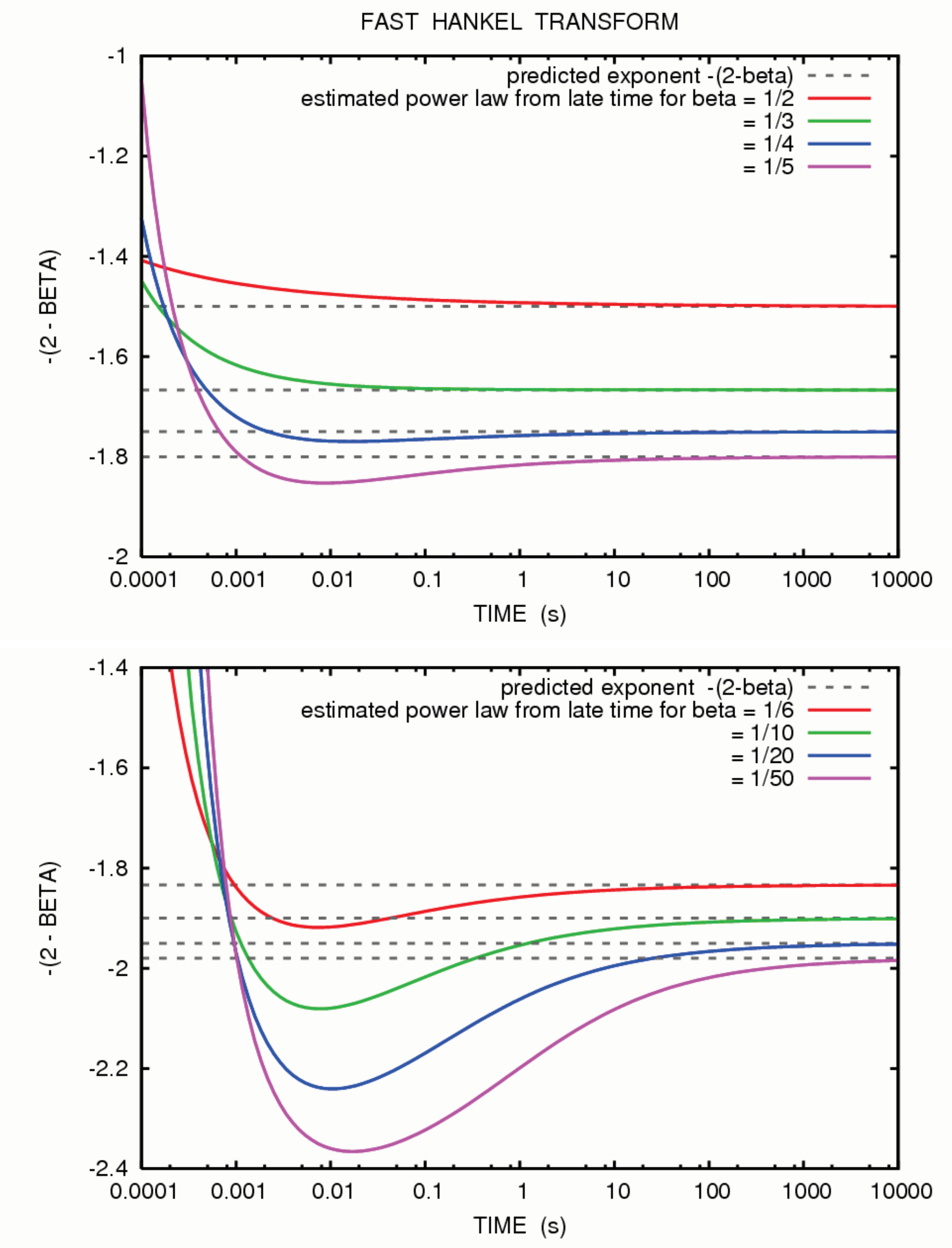

Figure 2: Numerical evaluation of the exponent in the transient decay and theoretical asymptotic limit for different roughness

Results

For the numerical experiment I used the same model as Everett, beside that here a vertical magnetic dipole instead of a horizontal circular loop is used. The base conductivity of the half-space is 0.1 S/m with combination of various roughness parameter β has been applied.

Figure 1 shows the transient as induced voltage due to an step response of the transmitter current for different roughness paramters. The large time range has been chosen to demonstrate the early and late time behavior in comparison with the asymptotic limits. For real measurements the time range will be much smaller. All transients - shown here - are calculated with the fast Hankel transform. As reference the transient of homogenous half-space with normal diffusion is also shown with grey lines. For separate loop configuration the transient shows a sign change. The negative values are shown with dashed lines and positive with solid lines. The asymptotic limits agree very well with theoretical prediced limits at eary and late times shown here as straight lines. The early time behaviour is for practical pupose of minor relevance, because in real meausrements the early time is influenced by system response of the system as the ramp for the current step off. So that in the data the straight line will not be visible. But the late time behaviour will be visible if the late time data can be measured and the data quatlity is sufficient.

To analyze how accurate the power law decay can be estimated from the calculated transients especialle for low roughness numbers, the transients are calculated up to extrem late times and the power law is determined by taking the numerical derivative *d ln H(t) / d ln t*. The result is shown in figure 2.

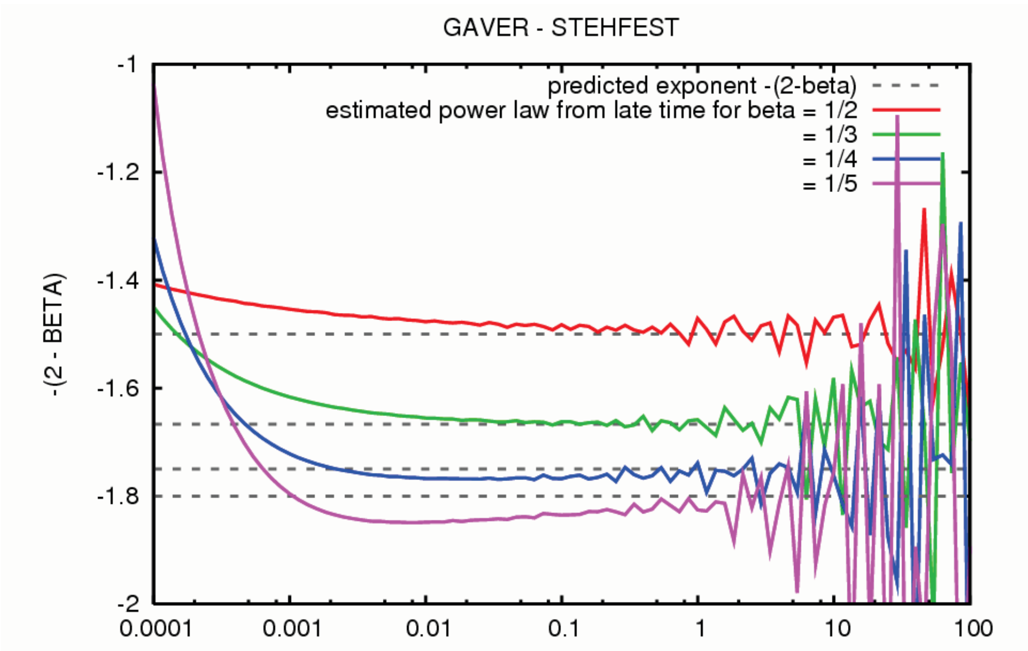

Figure 3: Numerical evaluation of the exponent in the transient decay and theoretical asymptotic limit for different roughness calculated with Gaver-Stehfest alogorithm

The graphs show the exellent accuracy of the fast Hankel transform, even for small roughness numbers, and indicates the smooth transition, when the roughness approaches zero. The influence of the anomalous diffusion moves to later times for decreasing roughness.
For very low roughness there will be a time range showing almost normal decay $t^{-5/2}$ and then goint to the predicted power law decay at extrem late time – approaching infinity. Responsible for this behavior is the second term of the low frequency approximation which is added.

The Gaver-Stehfest algorithm is not sufficient to estimate the power law decay. I tried different ways of programming, e.g to consider numerical accuracy the Laplace transform is done before the Hankel transform over the spatial wave number as recommended by Knight and Raiche (1982). The best results are shown in figure 3, achieved with 12 coefficients and using the analytical response for a vertical magnetic dipole in Lapace domain. Notice that the time range is shorter but still 3 decades more than in Everett's paper.

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