Electromagnetic induction in the spherical rotating earth due to asymmetric current loops or belts

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Abstract

The paper presents theoretical formulae for calculation of the spherical harmonic expansion of magnetic potential for current loop or belt model for stationary current systems in the high ionosphere and magnetosphere. The axis of symmetry for the current system does not coincide with the axis of Earth's rotation. Due to inclination of these axes there occurs azimuthal asymmetry of the exciting magnetic field which causes time-harmonic field for the observator on the rotating Earth. Theoretical EM field can be calculated by means of theory EM induction for the multilayered sphere for the superposition of spherical waves for discrete angular frequencies $m \cdot \Omega$, where m is order of the spherical harmonics and Ω is angular frequency of the Earth's rotation. The paper presents theoretical graphs of time variations of components (B_x, B_y, B_z) at selected observatory on the surface of the rotating Earth for near auroral (polar) or distant quasi equatorial current belts. There is shown that in the time course of magnetic field is dominant diurnal time period, corresponding to $m = 1$, while $m = 0$ corresponds to the steady external field.

Introduction

The advanced geomagnetic research of the Earth's space has discovered that in the Earth's ionosphere and magnetosphere there exist numerous huge electric current systems of complex geometry and time changes. The most known and closest to the Earth's surface is the ionospheric system generating the S_q geomagnetic variations *(Campbell, 1989)*, another current systems occur in the auroral oval *(Akasofu, 1972)* which cause the substorms, etc. Very important is the ring current system at the distances 2–5 R_e, which persists also in quiet magnetospheric state and during the perturbed solar wind conditions becomes the source of geomagnetic storms. This ring current is volume distributed, its intensity is dependent on both the distance from the Earth' centre and polar angle Θ. The analyses presented e.g. in *Nishida (1978)* show interesting property that the current intensity in the interior ring at $r \approx 3\,R_e$ is directed eastward (intensity about $I_1 \doteq +\,80000$ A) and in the outer oval at $r \approx 4.5\,R_e$ is current directed westward (intensity about $I_2 \doteq -\,1100000$ A). Using the Amper's law we can easily find that this westward current is clearly dominant during the main phase of the magnetic storm, since during this phase the horizontal (northward) component of the geomagnetic field on the middle latitudes strongly decreases.

The circular current loop/belt model

Let us consider the source of the external magnetic field the steady current spherical sheet of radius $a > R_e$. The surface current density $\boldsymbol{i} \equiv (0, i_\Theta, i_\Phi)$ as shows scheme in Fig. 1. The basic formulae for the exciting magnetic field we derived by using stream function Ψ concept according to *Smythe, 1950*, where the magnetic field components are derived by the vector potential $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{B}} = \nabla\times\hat{\boldsymbol{A}}$. The "hat" symbol denotes fields in the stationary (non-rotating) co-ordinates (r, Θ, Φ), firmly linked to the current source spherical sheet. We will use equivalent formulae for the magnetic field potential

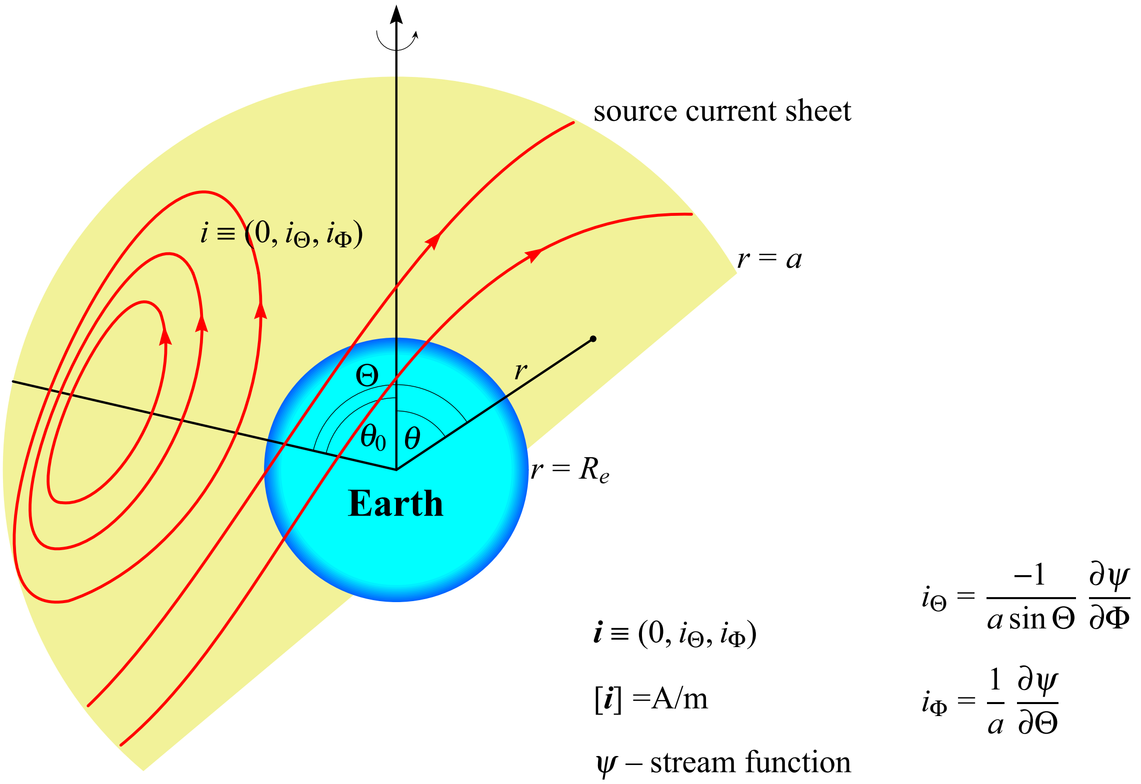


Fig. 1. Simplified spherical source current sheet

$\hat{V}(r, \Theta, \Phi)$ by putting $\hat{V} = -\partial(r\,W)/\partial r$ and $\hat{\boldsymbol{B}} = -\,\mathrm{grad}\ \hat{V}$. If the stream function $\Psi(\Theta, \Phi)$ is expressed by the sum of spherical harmonics $S_n^m(\Theta, \Phi)$:

$$\Psi(\Theta, \Phi) = \sum_{n=1}^{\infty} \sum_{m=0}^{n} S_n^m(\Theta, \Phi), \tag{1}$$

then the magnetic field potential outside of source sheet ($r > a$) is:

$$\hat{V}_o = -\mu_0 \sum_{n=1}^{\infty} \frac{n}{2n+1} \left(\frac{a}{r}\right)^{n+1} \sum_{m=0}^{n} S_n^m(\Theta, \Phi), \tag{2}$$

and in the interior ($r < a$):

$$\hat{V}_i = \mu_0 \sum_{n=1}^{\infty} \frac{n+1}{2n+1} \left(\frac{r}{a}\right)^{n} \sum_{m=0}^{n} S_n^m(\Theta, \Phi). \tag{3}$$

These potential are discontinuous on the source spherical sheet, there must be:

$$\mu_0 \Psi(\Theta, \Phi) = \left[\hat{V}_i - \hat{V}_o\right]_{r=a}. \tag{4}$$

For the zonal currents in the spherical sheet (current flow along paralells of co-latitude Θ) there is axial symmetry and we have:

$$\Psi(\Theta) = \sum_{n=1}^{\infty} c_n P_n(\cos\Theta). \tag{5}$$

Current density has only Φ component:

$$i_\Phi = \frac{1}{a} \frac{\partial \Psi}{\partial \Theta} = -\sum_{n=1}^{\infty} c_n P_n^1(\cos\Theta). \tag{6}$$

The potential of magnetic field which excites Earth $r < a$:

$$\hat{V}_i = \mu_0 \sum_{n=1}^{\infty} \frac{n+1}{2n+1} \left(\frac{r}{a}\right)^n c_n P_n(\cos\Theta). \tag{7}$$

In next explanations this potential we will denote as $\hat{V}^e$, since is external with respect to the Earth's body. As a simple model of the ring current is a circular loop of radius $a(> 3\,R_e)$ bearing electric current I, situated near the geomagnetic equatorial plane. The angle between the axis of current circle and axis of symmetry of the belt we denote as α. The axis of current loop/belt we consider running through the Earth's centre and inclined by the angle θ_0 to the earth's north semi axis. The situation is shown in the Fig. 2 and $c_n = -I(2n+1)\sin\alpha\, P_n^1(\cos\alpha)/[2n(n+1)]$.
The magnetic field components for the region $r < a$ due to this single loop in the r, Θ variables are known e.g. from *Smythe (1950)* in the form:

$$\hat{B}_r^e = \frac{\mu_0 I \sin\alpha}{2a} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^{n-1} P_n^1(\cos\alpha) P_n(\cos\Theta),$$

$$\hat{B}_\Theta^e = \frac{-\mu_0 I \sin\alpha}{2a} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{a}\right)^{n-1} P_n^1(\cos\alpha) P_n^1(\cos\Theta). \tag{8}$$

Here $P_n^m(\cos\Theta)$ are the Legendre functions degree n orders $m = 0, 1$. These formulae were derived from the magnetic vector potential with non zero azimuthal component A_ϕ. For the

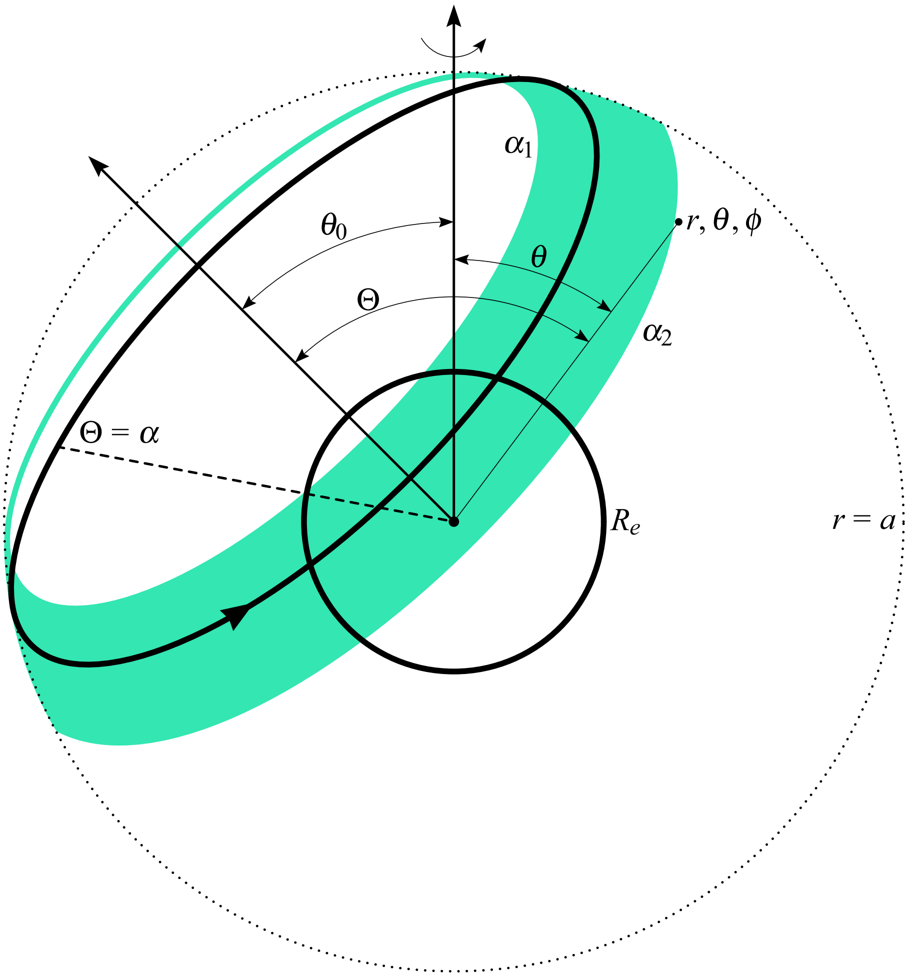


Fig. 2. The geometrical scheme of the current belt (green) at the sphere $r = a$ distributed at polar angle distances $\Theta \in \langle\alpha_1, \alpha_2\rangle$ above the spherical Earth. The black circle refers to the current loop.

geomagnetic purposes there is more suitable find the scalar potential which negative gradient gives the components (1). After some modifications we can find this potential in the form:

$$ {}^{e}\hat{V}(r,\Theta) = -R\sum_{n=1}^{\infty}(r/R)^n A_n P_n(\cos\Theta), \tag{9} $$

where $R = R_e$ is the radius of the Earth and coefficients A_n are:

$$ A_n = \frac{\mu_0 I \sin\alpha}{2a}\cdot\frac{1}{n}\left(\frac{R}{a}\right)^{n-1}\cdot P_n^1(\cos\alpha), \tag{10} $$

we can easily find that B_r and B_Θ are given by the derivatives:

$$ \hat{B}_r = -\frac{\partial\, {}^{e}\hat{V}}{\partial r}, \quad \hat{B}_\Theta = -\frac{1}{r}\frac{\partial\, {}^{e}\hat{V}}{\partial\Theta}, \tag{11} $$

since $P_n^1(\cos\Theta) = -\partial P_n(\cos\Theta)/\partial\Theta$.
From the principle of superposition for stationary electromagnetic field it is clear that potential (2) can be generalized to the system of current loops carrying the intensity I_k, their position is given by the support sphere radius a_k and polar angle α_k. Then we obtain from (3) formula for coefficients of the exciting potential

$$ \tilde{A}_n = \frac{\mu_0}{2n}\sum_{k=1}^{N}\frac{I_k\sin\alpha_k}{a_k}\cdot\left(\frac{R}{a_k}\right)^{n-1}\cdot P_n^1(\cos\alpha_k). \tag{12} $$

We can also transform this summation formula to the case of continuous distribution electric current density in the azimuthal direction $J_\phi(\Theta')$, which is distributed in the polar angle distances $\langle\alpha_1,\alpha_2\rangle$ above the sphere $r > R$. Then we will have:

$$ \tilde{A}_n = \frac{\mu_0}{2na}\left(\frac{R}{a}\right)^{n-1}\int_{\alpha_1}^{\alpha_2} J_\phi(\Theta')\sin\Theta' P_n^1(\cos\Theta')\,\mathrm{d}\,\Theta'. \tag{13} $$

The magnetic field potential in co-ordinates (r,Θ) can be easily transformed into stationary spherical system (r,θ,ϕ) with polar axis identical with Earth rotation axis. Let the θ,ϕ co-ordinate angles of north pole crossection of the current system axis symmetry are (θ_0,ϕ_0). Then the expression for the $\cos\Theta$ will be:

$$ \cos\Theta = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi-\phi_0). \tag{14} $$

Using the additional theorem for $P_n(\cos\Theta)$ *(e.g. Stratton, 1941)* we will have:

$$ \begin{aligned} P_n(\cos\Theta) = {} & P_n(\cos\theta)P_n(\cos\theta_0) + \\ & + 2\sum_{m-1}^{\infty}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta_0)P_n^m(\cos\theta)\cos m(\phi-\phi_0). \end{aligned} \tag{15} $$

The magnetic potential ${}^{e}\hat{V}$ will be real part of the complex expression:

$$ {}^{e}\hat{V}(r,\theta,\phi) = -R\sum_{n=1}^{\infty}(r/R)^n\sum_{m=0}^{n}C_{n,m}P_n^m(\cos\theta)\exp\left[-\mathrm{i}\, m(\phi-\phi_0)\right], \tag{16} $$

where the spherical harmonics coefficients C_n are calculated from A_n using relation:

$$ C_{nm} = A_n(2-\delta_{m,0})\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta_0), \tag{17} $$

$\delta_{m,0} = 1$ for $m = 0$, $\delta_{m,0} = 0$ for $m \geq 1$ is Kronecker's symbol. Here we use the associated Legendre functions in the form:

$$P_n^m(x) = (1-x^2)^{m/2} \frac{\mathrm{d}^m P_n(x)}{\mathrm{d}x^m}. \tag{18}$$

We will have e.g. $P_1^1(x) = \sqrt{1-x^2}$, $P_2^1(x) = 3x\sqrt{1-x^2}$, $P_2^2(x) = 3(1-x^2)$ etc. We can see, that each coefficient A_n generates a family of $n+1$ coefficients C_{nm}. The components of the exciting magnetic field can be easily calculated by means of $-\operatorname{grad} {}^e\hat{V}(r,\theta,\phi)$:

$${}^e\hat{B}_r = -\frac{\partial}{\partial r}({}^e\hat{V}),\quad {}^e\hat{B}_\theta = -\frac{1}{r}\frac{\partial}{\partial\theta}({}^e\hat{V}),\quad {}^e\hat{B}_\phi = -\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}({}^e\hat{V}). \tag{19}$$

Let us note that these formulae are more suitable for geomagnetic studies in comparison to the formulae using the elliptic integrals in the paper *Fuji and Schulz (2002)*.
The current density in the stationary system (r,θ,ϕ) we suppose as stationary, but for the observator (geomagnetic observatory) on the rotating Earth with geographical co-ordinates (r,θ,λ) the external field potential becomes time-harmonic, since the azimuthal function exp $\left[-\mathrm{i}\,m(\phi-\phi_0)\right]$ must be transformed into $\exp[-\mathrm{i}\,m(\lambda-\phi_0+\Omega t)]$, where Ω is angular frequency of Earth's rotation. This means that on the rotating Earth we observe asymmetric stationary external magnetic field as time-harmonic, with discrete angular frequencies $\omega = m\cdot\Omega$.
The problem of EM induction in the rotating sphere (Earth) can be solved by means of low-velocity relativistic electrodynamics (e.g.*Bullard, 1949; Sochelnikov, 1979*) the appropriate Maxwell equations in the non rotating reference frame are:

$$\nabla\times\hat{\boldsymbol{B}} = \hat{\boldsymbol{j}}\mu_0,\quad \nabla\times\hat{\boldsymbol{E}} = -\partial\hat{\boldsymbol{B}}/\partial t,\quad \nabla\cdot\hat{\boldsymbol{B}} = 0. \tag{20}$$

The density of the electric current $\hat{\boldsymbol{j}}$ is:

$$\hat{\boldsymbol{j}} = \begin{cases} \sigma(\hat{\boldsymbol{E}}+\hat{\boldsymbol{v}}\times\hat{\boldsymbol{B}}), & \text{for } r\leq R_e,\\ 0 & \text{for } r>R_e. \end{cases} \tag{21}$$

We can see that for the region of the rotating sphere (Earth), $r \leq R_e$ the magnetic field obeys equation:

$$\nabla\times\nabla\times\hat{\boldsymbol{B}} = -\sigma\mu_0\left[\partial\hat{\boldsymbol{B}}/\partial t - \nabla\times(\hat{\boldsymbol{v}}\times\hat{\boldsymbol{B}})\right]. \tag{22}$$

In the region outside the sphere the magnetic field satisfies equation

$$\nabla\times\hat{\boldsymbol{B}} = 0,\quad r>R_e, \tag{23}$$

so it can be calculated by gradient of the scalar potential in spherical functions of variables r,θ,ϕ. In our case we have only rotational motion of the conducting sphere around the polar axis $\theta = 0$ of the co-ordinate system $\hat{S}$, so the velocity vector will have only ϕ-component:

$$\hat{\boldsymbol{v}} \equiv (0,0,v_\phi),\quad \hat{v}_\phi = \Omega r\sin\theta. \tag{24}$$

Careful calculation of the expression $\nabla\times(\hat{\boldsymbol{v}}\times\hat{\boldsymbol{B}})$ will give:

$$\nabla\times(\hat{\boldsymbol{v}}\times\hat{\boldsymbol{B}}) = -\Omega\partial\hat{\boldsymbol{B}}/\partial\phi. \tag{25}$$

Considering that in our non rotating reference frame we have $\partial\hat{\boldsymbol{B}}/\partial t = 0$, we obtain from (22) equation:

$$\nabla\times\nabla\times\hat{\boldsymbol{B}} + \sigma\mu_0\Omega\partial\hat{\boldsymbol{B}}/\partial\phi = 0. \tag{26}$$

The magnetic field potential (16) due to stationary electric currents depends on ϕ co-ordinate via $\exp[-\mathrm{i}\, m(\phi - \phi_0)]$, where $m = 0, 1, 2, \dots$ is (azimuthal) order number of spherical harmonics. The same dependence will be transmitted to the magnetic field, then the derivative with respect to ϕ for m-harmonics in equation (26) can be easily performed and we obtain for m-th harmonics ${}_m\hat{\boldsymbol{B}}$ from (26):

$$\nabla \times \nabla \times ({}_m\hat{\boldsymbol{B}}) = \mathrm{i}\, \mu_0 m \sigma \Omega ({}_m\hat{\boldsymbol{B}}). \tag{27}$$

For the observer (observatory) on the rotating Earth with co-ordinates (r, θ, λ); θ is the geographical co-latitude angle and λ is the geographical longitude angle, we must put transformation:

$$\phi = \lambda + \Omega t, \tag{28}$$

where t denotes UT. The magnetic field on the surface or inside of rotating Earth we denote $\boldsymbol{B} \equiv (B_r, B_\theta, B_\lambda)$. It will be time harmonic dependence, since from (26) we obtain for its m-harmonics ${}_m\boldsymbol{B}$ equation:

$$\nabla \times \nabla \times ({}_m\boldsymbol{B}) = \mathrm{i}\, \omega \mu_0 \sigma ({}_m\boldsymbol{B}), \tag{29}$$

where $\omega = m\Omega$. This is well known EM induction equation for angular frequency ω, which can be solved by standard treatment using spherical wave functions for poloidal type of magnetic field.

Time harmonic EM induction in the multilayered rotating Earth

The problem of the time-harmonic excitation field and its induction effect in the sphere has been gradually developed during last century e.g.: Lamb, Schuster, Chapman, Rikitake, Lahiri, Price. Very detailed monographies in this topic is are *(Rotanova and Pushkov 1982; Berdichevsky and Zhdanov, 1984; Campbell, 1987)*. Valuable knowledge to the basic EM induction problem for the spherical Earth can be also found in papers *(Pěč, Martinec and Pěčová, 1985)* and more recently in *(Maus and Lühr, 2005)* as well as *(Velímský and Martinec, 2005; Velímský, Martinec and Everett, 2006)*. In our study we present formulae which use more suitable form of spherical Bessel functions, which simplifies expressions for reflection and transmission coefficients on spherical boundaries. The harmonic time variability of the exciting potential ${}^eV(r, \theta, \lambda)$ we suppose in the form $\exp(-\mathrm{i}\, \omega t)$, where $\omega = m\Omega$ and introduce the complex potential:

$${}^eU(r, \theta, \lambda) = -R \sum_{n,m} C_{nm} (r/R)^n P_n^m(\cos\theta)\, G_m(\lambda, t), \tag{30}$$

where $G_m(\lambda, t) = \exp[-\mathrm{i}\, m(\lambda - \phi_0 + \Omega t)]$.
Physical reality we assign to the real part of the ${}^eU(r, \theta, \lambda)$ and its gradient. The individual components of the exciting magnetic field are of individual spherical harmonics (n, m) are:

$$\begin{aligned}
{}^e_rB_{nm} &= {}^0C_{nm}(r/R)^{n-1} \cdot nP_n(\cos\theta)\, G_m(\lambda, t) \\
{}^e_\theta B_{nm} &= {}^0C_{nm}(r/R)^{n-1} \cdot \frac{\mathrm{d}\, P_n^m(\cos\theta)}{\mathrm{d}\,\theta} G_m(\lambda, t) \\
{}^e_\lambda B_{nm} &= {}^0C_{nm}(r/R)^{n-1} \cdot \frac{P_n^m(\cos\theta)}{\sin\theta} \frac{\partial G_m(\lambda, t)}{\partial \lambda}.
\end{aligned} \tag{31}$$

The Earth we consider as a spherical multilayered of radius $R (= R_e)$, consisting of L concentric spherical layers till the core mantle boundary at the depth 2900 km, the core $(r \le r_L)$ is considered as a uniform sphere of conductivity σ_L. We introduce $r_1 = R_e$ and the conductivity σ_j is assumed constant in the layer $r_{j+1} \le r \le r_j$. The magnetic permeability is assumed constant in all layers, as well in the non-conducting region "0" outside the sphere and equal $\mu_0 = 4\pi \times 10^{-7}$ Henry/m. The slowly time varying EM field in the individual layers obeys the Maxwell's equations

$$\nabla \times \boldsymbol{B} = \sigma\mu_0\boldsymbol{E}, \quad \nabla \times \boldsymbol{E} = +\mathrm{i}\,\omega\boldsymbol{B}. \tag{32}$$

The magnetic induction $\boldsymbol{B}$ now represents the time-harmonic field of selected angular frequency ω. In a view of (28) and (31) this field obeys the vector wave equation

$$\nabla \times \nabla \times \boldsymbol{B} = \mathrm{i}\,\omega\mu_0\sigma\boldsymbol{B} \tag{33}$$

It solution in spherical co-ordinate system does not reduce to the scalar Helmholtz equation as in the Carthesian co-ordinates. The $\boldsymbol{B}$ vector must be separated into toroidal and poloidal parts as is proved in details by *Stratton, 1941; Born and Wolf, 1964*. If the exciting field is poloidal like (31), the induced magnetic field in a spherically concentric layers will be poloidal too and its (r, θ, λ) components will have the same θ, λ dependence as in (31). Their radial dependence will be expressed by the spherical functions $f_n = \psi_n(kr)$, $\zeta_n(kr)$ which obey the ordinary differential equation

$$f_n''(z) + \left[1 - n(n+1)/z^2\right] f_n = 0, \quad z = kr. \tag{34}$$

It was proved in *Born, Wolf, 1964* that the solutions are functions:

$$\psi_n(kr) = (\pi kr/2)^{1/2} J_{n+1/2}(kr), \quad \zeta_n(kr) = (\pi kr/2)^{1/2} H^{(1)}_{n+1/2}(kr), \tag{35}$$

where $J_{n+1/2}(kr)$, $H^{(1)}_{n+1/2}(kr)$ are Bessel function and Hankel function of the first kind half integer index $n + 1/2$ and complex argument kr:

$$kr = r(\mathrm{i}\,\omega\sigma\mu_0)^{1/2} = r(1+\mathrm{i})(\omega\sigma\mu_0/2)^{1/2}. \tag{36}$$

In the j-th layer this argument will carry the dependence on the electrical conductivity σ_j, this means:

$$k_j r = r(1+\mathrm{i})(\omega\sigma_j\mu_0/2)^{1/2}. \tag{37}$$

The spherical (n, m) mode of the magnetic field will have in the j-th layer components:

$$\begin{aligned}
{}^j_rB_{nm} &= \left[{}^jC_{nm}\psi_n(k_jr) + {}^jD_{nm}\zeta_n(k_jr)\right] n(n+1)(k_jr)^{-2} P_n^m(\cos\theta) G_m(\lambda, t),\\
{}^j_\theta B_{nm} &= \left[{}^jC_{nm}\psi_n'(k_jr) + {}^jD_{nm}\zeta_n'(k_jr)\right] (k_jr)^{-1}\, \mathrm{d}\,P_n^m(\cos\theta)/\,\mathrm{d}\,\theta\, G_m(\lambda, t),\\
{}^j_\lambda B_{nm} &= \left[{}^jC_{nm}\psi_n'(k_jr) + {}^jD_{nm}\zeta_n'(k_jr)\right] (k_jr)^{-1} P_n^m(\cos\theta)/\sin\theta\,\partial G_m(\lambda, t)/\partial\lambda.
\end{aligned} \tag{38}$$

In the bottom sphere $r \le \mathrm{d}_L$ (the Earth's core) we must put ${}^LD_{nm} \equiv 0$, since in this region $kr \to 0$, where $\zeta_n(kr)$ and its derivative is singular. In the non-conducting region $r \in \langle R, a)$ we will have secondary magnetic field with potential

$${}^sU(r, \theta, \lambda) = -R \sum_{n,m} (R/r)^{n+1} \cdot {}^0D_{nm} P_n^m(\cos\theta) G_m(\lambda, t). \tag{39}$$

Pertinent components for this potential can be expressed by negative grad ${}^sU(r, \theta, \lambda)$. From the EM theory we know that on each conductivity spherical boundary must be continuous transition of radial (rB_{nm}) and tangentional (${}_\theta B$ or ${}_\lambda B_{nm}$) components. In this manner we obtain the system of linear equations for unknown coefficients ${}^jC_{nm}, {}^jD_{nm}$ while only the exciting field coefficients ${}^0C_{nm}$

we consider to be known (calculated for the external exciting electric currents). We shall briefly give algorithm from our previous papers *Hvoždara, 1976, 1980*. Then we have: for $r = r_1 = R$:

$$n\,{}^0C_{nm} - (n+1)\,{}^0D_{nm} = \left[{}^1C_{nm}\psi_n(k_1r_1) + {}^1D_{nm}\zeta_n(k_1r_1)\right] n(n+1)(k_1r_1)^{-2},$$

$${}^0C_{nm} + {}^0D_{nm} = \left[{}^1C_{nm}\psi_n'(k_1r_1) + {}^1D_{nm}\zeta_n'(k_1r_1)\right](k_1r_1)^{-1}. \tag{40}$$

On the boundary $r = r_j$, ($j \neq 1$, $j \neq L$):

$${}^{j-1}C_{nm}\psi_n(k_{j-1}r_j) + {}^{j-1}D_{nm}\zeta_n(k_{j-1}r_j) = \gamma_{j-1}^2\left[{}^jC_{nm}\psi_n(k_jr_j) + {}^jD_{nm}\zeta_n(k_jr_j)\right]$$

$${}^{j-1}C_{nm}\psi_n'(k_{j-1}r_j) + {}^{j-1}D_{nm}\zeta_n'(k_{j-1}r_j) = \gamma_{j-1}\left[{}^jC_{nm}\psi_n'(k_jr_j) + {}^jD_{nm}\zeta_n'(k_jr_j)\right] \tag{41}$$

where $\gamma_{j-1} = (\sigma_{j-1}/\sigma_j)^{1/2}$. For the deepest boundary $r = d_L$ (surface of the core) we have:

$${}^{L-1}C_{nm}\psi_n(k_{L-1}d_L) + {}^{L-1}D_{nm}\zeta_n(k_{L-1}d_L) = \gamma_{L-1}^2\,{}^LC_{nm}\psi_n(k_Ld_L)$$

$${}^{L-1}C_{nm}\psi_n'(k_{L-1}d_L) + {}^{L-1}D_{nm}\zeta_n'(k_{L-1}d_L) = \gamma_{L-1}\,{}^LC_{nm}\psi_n'(k_Ld_L). \tag{42}$$

This system of equations is solvable by the elimination methods, because equations for $r = r_2, r_3, \ldots, r_L$ are homogeneous. The we introduce the proportionality ${}^jW_n$ and reflection coefficients ${}^jF_n$ as follows:

$${}^LW_n = \psi_n'(k_Lr_L)/[\gamma_{L-1}\psi_n(k_Lr_L)], \quad {}^{L-1}D_{nm} = (-1)\cdot{}^{L-1}F_n\cdot{}^{L-1}C_{nm}, \tag{43}$$

where:

$${}^{L-1}F_n = \frac{\psi_n'(k_{L-1}r_L) - {}^LW_n\,\psi_n(k_{L-1}r_L)}{\zeta_n'(k_{L-1}r_L) - {}^LW_n\,\zeta_n(k_{L-1}r_L)}.$$

On the boundaries $r = d_{L-1}, \ldots, d_2$ we have similarly:

$${}^jW_n = \frac{\psi_n'(k_jr_j) - {}^jF_n\,\zeta_n'(k_jr_j)}{\gamma_{j-1}[\psi_n(k_jr_j) - {}^jF_n\,\zeta_n(k_jr_j)]}. \tag{44}$$

$${}^{j-1}D_{nm} = (-1)\cdot{}^{j-1}F_n\cdot{}^{j-1}C_{nm}, \tag{45}$$

$${}^{j-1}F_n = \frac{\psi_n'(k_{j-1}r_j) - {}^jW_n\,\psi_n(k_{j-1}r_j)}{\zeta_n'(k_{j-1}r_j) - {}^jW_n\,\zeta_n(k_{j-1}r_j)}, \tag{46}$$

In fact the general expressions (44)–(46) also include the interface $r = d_L$, it should be put ${}^LF_n \equiv 0$. Now we have to consider equations (40) relevant to the boundary $r = r_1 (\equiv R)$ which can be written in the form

$$(n+1)\,{}^0D_{nm} + {}^1C_{nm}[\psi_n(k_1r_1) - {}^1F_n\,\zeta_n(k_1r_1)]\,n(n+1)\,(k_1r_1)^{-2} = n\,{}^0C_{nm},$$

$$-{}^0D_{nm} + {}^1C_{nm}[\psi_n'(k_1r_1) - {}^1F_n\,\zeta_n'(k_1r_1)]/(k_1r_1) = {}^0C_{nm}. \tag{47}$$

From this system of two linear equations we can easily find that the

$${}^1C_{nm} = \frac{(k_1r_1)\,(2n+1)\,{}^0C_{nm}}{(n+1)[\psi_{n-1}(k_1r_1) - {}^1F_n\,\zeta_{n-1}(k_1r_1)]}, \tag{48}$$

$${}^0D_{nm} = \frac{-n\,{}^0C_{nm}[\psi_{n+1}(k_1r_1) - {}^1F_n\,\zeta_{n+1}(k_1r_1)]}{(n+1)[\psi_{n-1}(k_1r_1) - {}^1F_n\,\zeta_{n-1}(k_1r_1)]}. \tag{49}$$

In derivation of these equations we have used recurrence properties of the radial functions:

$$f_n'(z) + (n/z)f_n(z) = f_{n-1}(z), \quad (2n+1)f_n(z) - zf_{n-1}(z) = zf_{n+1}(z), \tag{50}$$

where $f_n(z) = \psi_n(z)$ or $\zeta_n(z)$. In this manner we can calculate the electromagnetic response function for our multilayered spherical Earth:

$$\frac{{}^0D_{nm}}{{}^0C_{nm}} = \frac{-n}{n+1}\,\frac{\psi_{n+1}(k_1r_1) - {}^1F_n\zeta_{n+1}(k_1r_1)}{\psi_{n-1}(k_1r_1) - {}^1F_n\zeta_{n-1}(k_1r_1)} = {}^0F_n. \tag{51}$$

By using the reccurrence relations (50) we can obtain alternative expression for ${}^0F_n$ in the form:

$${}^0F_n = \frac{n}{n+1}\left\{1 - \frac{(2n+1)[\psi_n(k_1r_1) - {}^1F_n\zeta_n(k_1r_1)]}{k_1r_1[\psi_{n-1}(k_1r_1) - {}^1F_n\zeta_{n-1}(k_1r_1)]}\right\}. \tag{52}$$

This is more suitable for numerical calculations since it contains the radial functions of neighbouring indices n and $n-1$. In the classical geomagnetic EM induction theory the coefficient ${}^0F_n$ corresponds to the ratio I_n/E_n, where I_n is the amplitude factor of the scalar potential of induced (interior) field and E_n is amplitude of the inducing (exciting) field for the spherical harmonics degree n. The coefficients E_n, I_n correspond to our ${}^0C_{nm}$, ${}^0D_{nm}$ respectively. We can see that by using the coefficient ${}^0F_n$ the amplitudes of radial and tangential components of the n-th ($m = 0$) harmonics have on the surface of the Earth are:

$${}_rb_n = [n - (n+1){}^0F_n]E_n, \quad {}_\theta b_n = [1 + {}^0F_n]E_n. \tag{53}$$

Their ratio can be used also for calculation of the surface impedance using *Berdichevsky and Zhdanov, (1984)* formulae:

$$Z_n = \frac{-\mathrm{i}\,\omega\mu_0R_e}{n(n+1)}\,\frac{{}_rb_n}{{}_\theta b_n} = \frac{-\mathrm{i}\,\omega\mu_0R_e}{n(n+1)}\,\frac{n-(n+1)\,{}^0F_n}{1+{}^0F_n}. \tag{54}$$

In this manner our formulae link to the global magnetovariational theory. Let us stress that EM response coefficients ${}^0F_n$ are independent of azimuthal number m. The same holds true for ratio ${}_rb_n/{}_\theta b_n$ for the concentric layered Earth conductivity model.

Numerical calculation of EM response functions and apparent resistivity

Recently we have prepared fast and reliable computer FORTRAN code for calculation of EM response coefficients ${}^0F_n$ for wide interval of time periods T, ranging from 3 hours till 6 years, using T_s as period T in seconds. These calculations were performed for various known depth-conductivity Earth's models, i.e. for $L \geq 5$ till $L = 12$. According to present knowledge, for the spherical Earth there must be considered two kinds of the crust and upper mantle electrical conductivity distribution models: continental (A), oceanic (B) ones.

The **continental model** is characterized by the superficial layer "1" of conductivity $\sigma_1 \approx 0.002$–0.02 S/m, thickness 8–15 km, then the conductivity σ_2 is about $\sigma_1/10$, because of dehydratation of rocks, its thickness is about 30 km. From the bottom of continental crust in depths ~ 30–50 km the conductivity grows due to increasing temperature and attains about 0.05 S/km in the continental astenosphere. In the mantle there is gradual growth of σ, attaining about 2–10 S/m on the mantle-core boundary in the depth 2900 km. Due to phase transitions of yhe mantle minerals and also growing temperature there are known boundaries with increase of electrical conductivity at depths about 410–4500 km, 800–1000 km, 1500–1800 km. In the Earth's core the electrical conductivity of hot Fe-Ni-S melt we consider to be uniform: $\sigma_L = 5000$ S/m. It means, that on this boundary

we have the ratio $\sigma_L/\sigma_{L-1} \approx 10^3$, which enables us to confine only on the calculation of transition coefficients ${}^jF_n$ in the layers in the mantle and crust, i.e. $j = L-1, \ldots, 1$ where the $r \in \langle r_c, R_E \rangle$, while $r_c = 0.5453 \times R_E$.
The **oceanic model** (B) is characterized by the sea water layer of the high conductivity $\sigma_1 \approx 0.4$–0.6 S/m, its mean thickness is about 4 km, which causes high attenuation of EM field of periods shorter than ~60 min. The rocks layers below the oceanic bottom have as a rule higher electrical conductivity in comparison with the continental ones, since the temperature and temperature gradient is higher about 20% in comparison to the continental litosphere.
The numerical calculations were tested for numerous adequate models, but here we present results for the "continental" model shown in Fig. 3a. The moduli and phases of ${}^0F_n$ for this model are in Fig. 3b together with apparent resistivity curves. The periods for one day and its fractions T/m

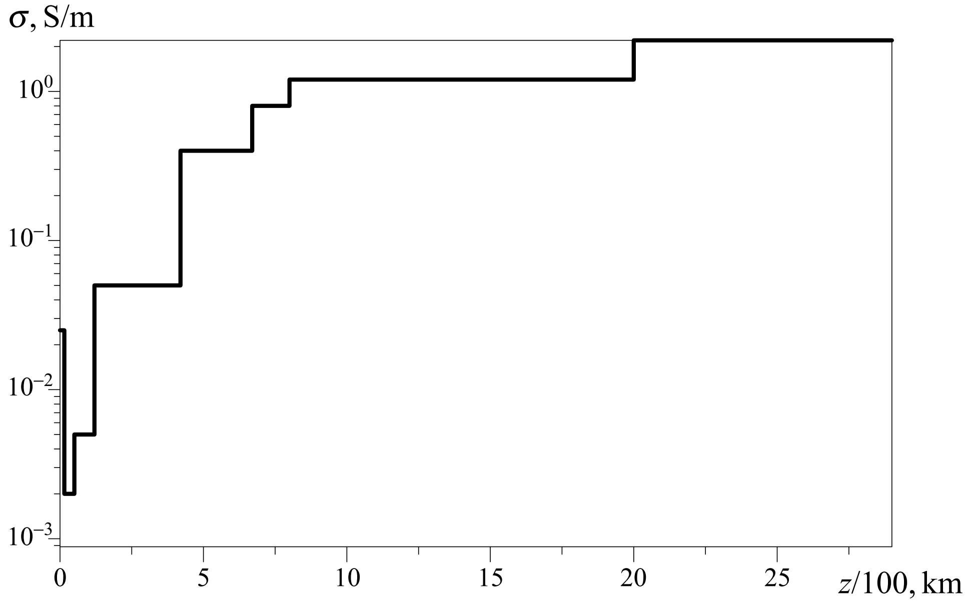
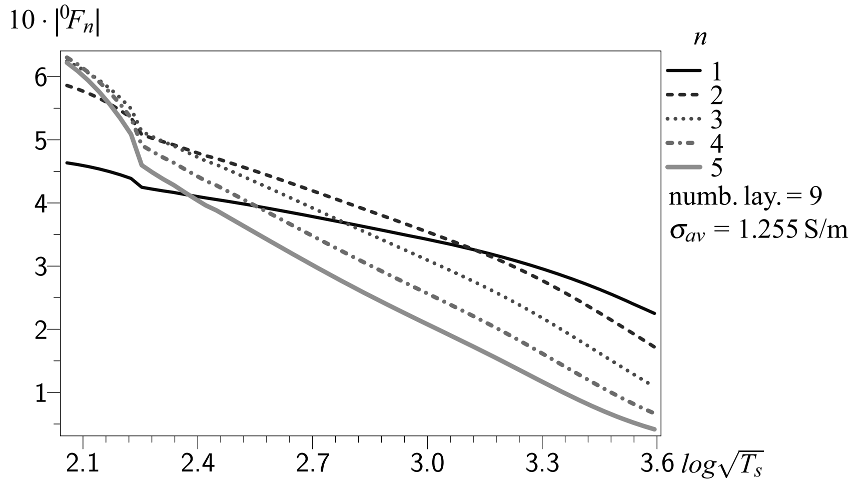


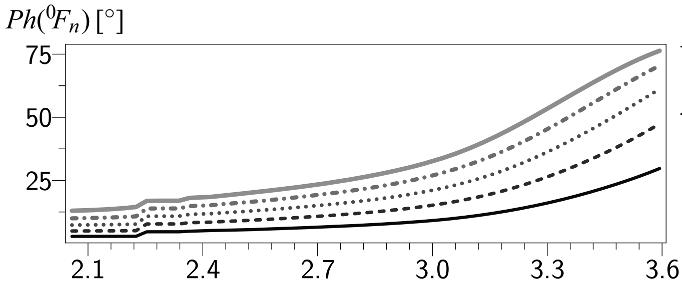
Fig. 3a. The graph of conductivity depth distribution in the Earth's crust and mantle (the "continental model„) as function of depth (z), used in present study.

can be found at the range $\log\sqrt{T_s} \approx 2.1$ as shows also the Tab. 1.

m	1	2	3	4	5
T/m, day	1	0.5	0.333	0.25	0.2
$\log\sqrt{T_s}$	2.468	2.317	2.229	2.167	2.118

For better resolution we plotted moduli of ${}^0F_n$ multiplied by the factor 10 and the apparent resistivity values were normed to $\rho_1 = (\sigma_1)^{-1}$ (the resistivity of the first layer). We can see, that the values of $|{}^0F_n|$ as a rule drop with $\sqrt{T_s}$ and the decrease of these values is steeper for increased degree n of spherical harmonics. The phases of ${}^0F_n$ attain values from 0° till ~ 75° and the phase shift grows with degree number n and period T_s. The values of $\log(\rho_a/\rho_1)$ decrease almost linearly with $\log\sqrt{T_s}$ and the curve for $n = 1$ is above those for $n \geq 2$, which is in agreement with results of *Berdichevsky and Zhdanov (1984)*. Let us note, that for calculations of ${}^0F_n$ for shorter period (less than 0.5 day) and high conductive layer $j = 1$ there must be used large value asymptotics of $\psi_n(z)$, $\zeta_n(z)$.





j	h_j [km]	σ_j S/m
1	0	0.0250
2	15	0.0020
3	50	0.0050
4	120	0.0500
5	420	0.4000
6	670	0.8000
7	800	1.200
8	2000	2.200
9	2900	5000.

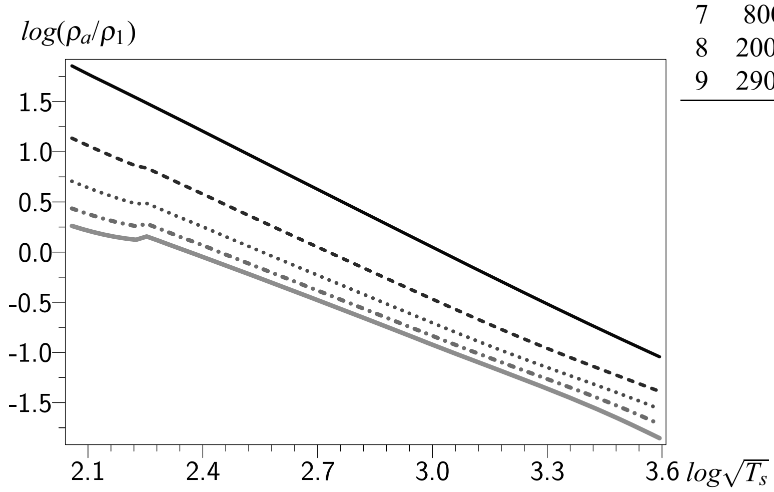


Fig. 3b. The graphs of calculated courses of EM response coefficients ${}^0F_n$, their phases and course of $\log(\rho_a/\rho_1)$ in dependence on $\log\sqrt{T_s}$, where T_s is period T in seconds. The pertinent values of depths boundaries h_j and conductivity values σ_j are given in the table, σ_{av} average conductivity for each model (considering depths to 2900 km).

Magnetic field due to some current belts

The calculations of the primary field coefficients $\tilde{A}_n$ were performed according to formula (13). These $\tilde{A}_n$ we multiply by spherical harmonics for necessary degree numbers n and $m = 0, 1, 2, \ldots 5$ and than using formula (17) we calculate coefficients C_{nm} for the exciting potential. By summation with respect to n and m we obtaine the magnetic field components. In order to use common local geomagnetic horizontal and vertical components on the surface of the Earth, we put:

$$B_x = -B_\theta, \quad B_y = B_\lambda, \quad B_z = -B_r. \tag{55}$$

The results of numerical calculations we present for the "continental" Earth conductivity model and three types of current belt: a) auroral, b) equatorial, c) Sq current model.

For belts a) and b) we put angular width 20° around central circle α, it means $\alpha_1 = \alpha - 10°$, $\alpha_2 = \alpha + 10°$. The axis of the belt we put at spherical coordinates $\theta_0 = 10°$, $\phi_0 = -110°$, which corresponds to North geomagnetic pole. The net intensity current we put $I = 10^7$ A, we suppose it as distributed uniformly on the width of the belt i.e. $J(\Theta') = I/(\alpha_2 - \alpha_1)$. The field components B_x, B_y, B_z we normed by common norming value $B_n = \mu_0 I/(2a_c)$, where a_c is the radius of the belt supporting sphere ($\equiv a$) in theoretical formulae.
For the **auroral belt** we chosed $\alpha = 20°$ and small distance from the Earth surface so $a_c/R_e = 1.1$ which means that this belt is in the hight $h_c = 636$ km and then $B_n = \mu_0 I/(2a_c) = 895.58$ nT. The theoretical time courses of the geomagnetic field components were calculated for the observation points: ($\theta = 30°$, $\lambda = 0°$), ($\theta = 42°$, $\lambda = 0°$), on the rotating Earth, which correspond to high-latitude (e.g. Nurmijarvi) or mid-latitude geomagnetic observatory, e.g. Hurbanovo. In Figs 4a,b there is shown the time course of the summary field (exciting + induced) during 46 hours. In both figures we can see dominant 24 h variation in all three components. The time course of horizontal components (B_x, B_y) resambles repeated geomagnetic variations with prevailing period 24 h. The time course of the exciting magnetic field on the rotating earth for $\theta = 42°$, $\lambda = 0°$ is presented in Fig. 4c. When comparing with Fig. 4b we can see that tangential components (B_x, B_y) are amplified due to induction, but vertical component B_z is attenuated. Theoretically we can see it in formula (53), where the n-th harmonics of tangential components on the surface are given by the terms $(1 + {}^0F_n)E_{nm}$, but in the radial component we have terms $[n - (n+1)^0F_n]E_{nm}$.
For the **equatorial belt** we chosed $\alpha = 90°$ and large distance from the Earth surface so $a_c/R_e = 3.0$, so this belt is in the hight $h_c = 12755$ km and then $B_n = \mu_0 I/(2a_c) = 328.38$ nT. The theoretical time courses of the geomagnetic field components were calculated for the same observation point (θ, λ) as in previous case. This equatorial belt we consider as a plausible model for the ring current.
In Fig. 5a there is shown time course of exciting (external) field and in Fig. 5b the summary field (exciting + induced) during 46 hours. In both figures we can see also dominant 24 h variation in all three components, but the amplitudes of diurnal waves are smaller in comparison with auroral belt. Numerical calculations proved that EM induction amplifies both horizontal components about 20% and strongly attenuates the vertical component, but the time variations on $\theta = 42°$ are almost the same as for $\theta = 30°$. The amplitudes of stationary primary field due to equatorial current belt along the meridian $\phi = 0°$ are presented in Fig. 5c. We can see that in this field there is dominant spherical harmonics $n = 1$, so B_x is proportional to $\sin\theta$, while B_z is proportional to $-\cos\theta$. Let us note, that in our calculations we consider the direction of stationary current as positive (Eastward)., while in the quiet real magnetospheric current the direction is opposite (Westward). The same holds true also for the disturbed Dst ring current. In some magnetograms of very strong geomagnetic storms as recorded at geomagnetic observatory Hurbanovo (e.g. during days 07–11 November, 2004) there is clearly present some part of the disturbed field as repeating with period one day.
Approximation of Sq current system on the northern hemisphere was considered as the current belt with axis of symmetry in the pole $\theta_0 = 45°, \phi_0 = 180°$ in order to meet knowledge given in textbooks *Parkinson, 1983* or *Campbell, 1989*. Very important feature in the time course of Sq geomagnetic variations is their uniform harmonic dependence on the local time and non-uniform distribution on co-latitude θ. There exist also differences for continents and seasons of the year, but these are not so pronounced. The time variations in local time t^* for various meridians λ we can consider as almost the same as along the meridian $\lambda = 0°$ where we use time t as UT. Simple calculation, using 1 hour as a unit for time, will show: $t^* = t + \lambda/(15°/\mathrm{h})$, [h], since the angular speed Ω of the Earth's rotation is: $\Omega = 360°/(24\mathrm{h}) = 15°/\mathrm{h}$. Then we will have for $\lambda - t$ expressions of the magnetic field: $m[\lambda - \phi_0 + \Omega t] = m[\Omega t^* - \phi_0]$. According to *Parkinson, 1983* the focus of the Sq current system in the ionosphere occurs at noon $t^* = 12$h (LT), so we must put $\phi_0 = 180°$,

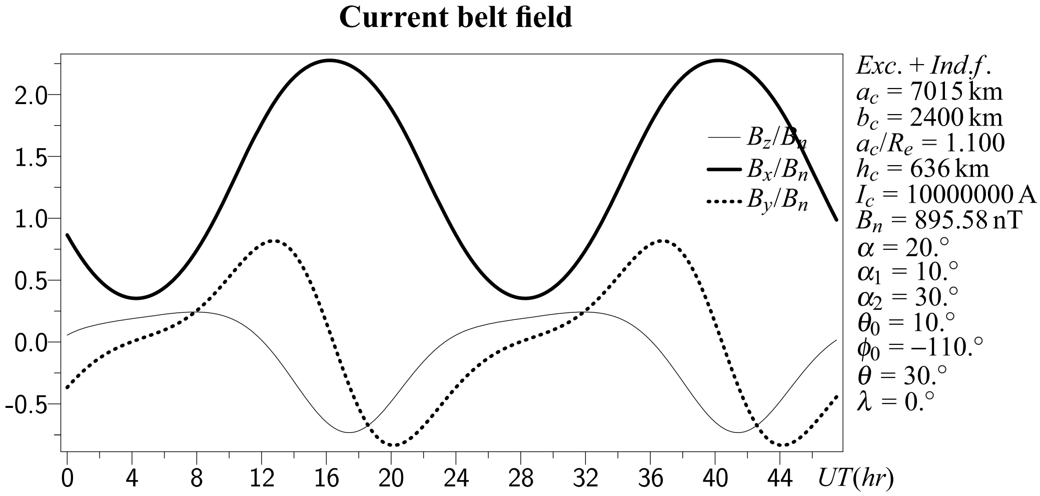


Fig. 4a. Time variations of summary (exciting + induced) surface magnetic field for the auroral current belt around central circle $\alpha = 20°$ near the North magnetic pole (θ_0, ϕ_0). The point observation is $\theta = 30°$, $\lambda = 0°$ on the rotating Earth.

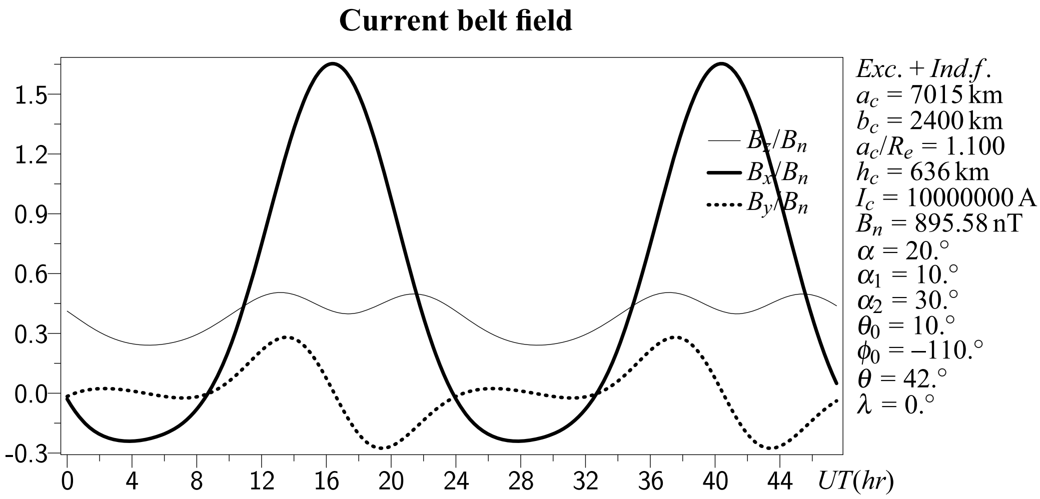


Fig. 4b. The same as in Fig. 4a, but for co-latitude $\theta = 42°$.

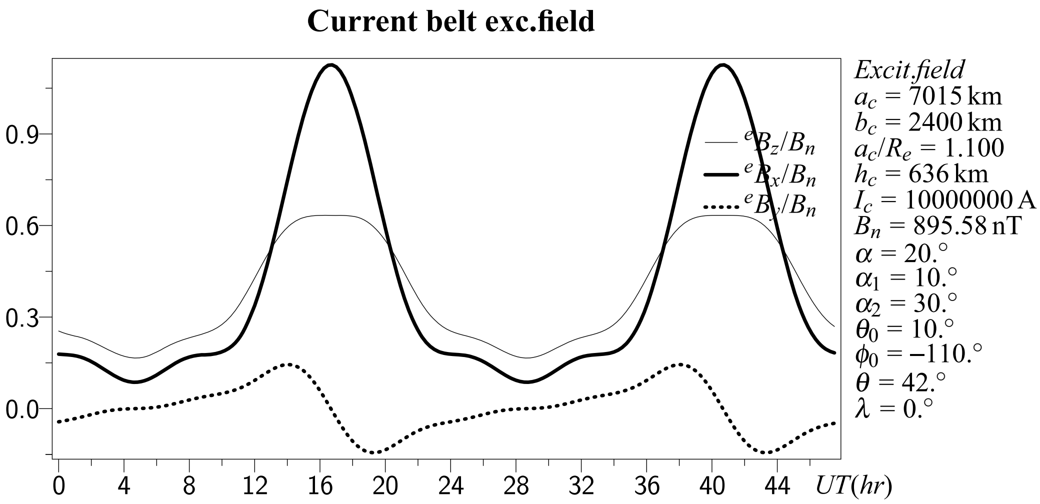


Fig. 4c. Time variations of the exciting field due to stationary auroral current belt ($\alpha_1 = 10°$, $\alpha_2 = 30°$) at the observatory $\theta = 42°$ meridian $\lambda = 0°$ on the rotating Earth.

Current belt field

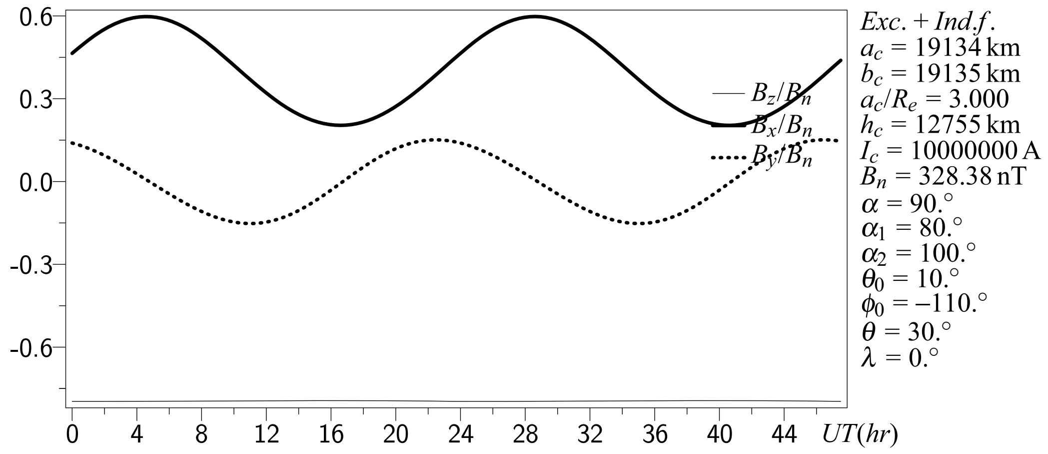


Fig. 5a. Time variations of summary (exciting + induced) surface magnetic field for the equatorial current belt around central circle $\alpha = 90°$ with central axis near the North magnetic pole (θ_0, ϕ_0). The point observation is $\theta = 30°$, $\lambda = 0°$ on the rotating Earth.

Current belt field

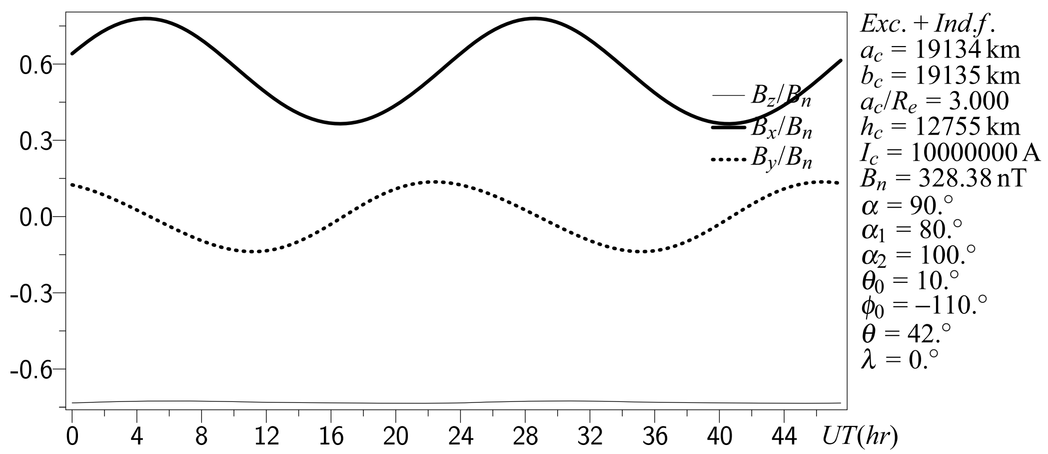


Fig. 5b. The same as in Fig. 5a, but for co-latitude $\theta = 42°$.

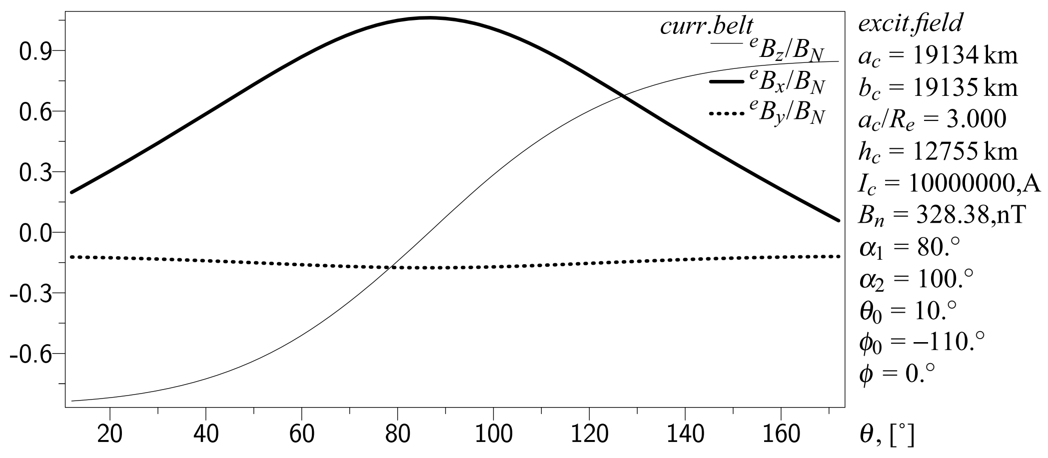


Fig. 5c. Amplitudes of the statitonary exciting field due to equatorial current belt along the meridian $\phi = 0°$ (in the non-rotating co-ordinate system).

Current belt field

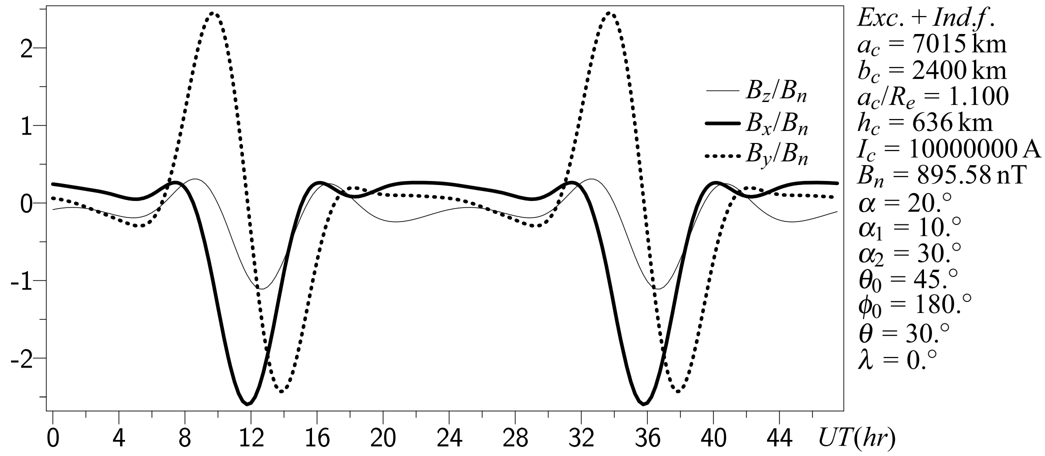


Fig. 6a. Time variations of summary (exciting + induced) surface magnetic field for the Sq current belt around central circle α = 20° with central axis at (θ_0 = 45°, ϕ_0 = 180°). The point observation is θ = 30°, λ = 0° on the rotating Earth.

Current belt field

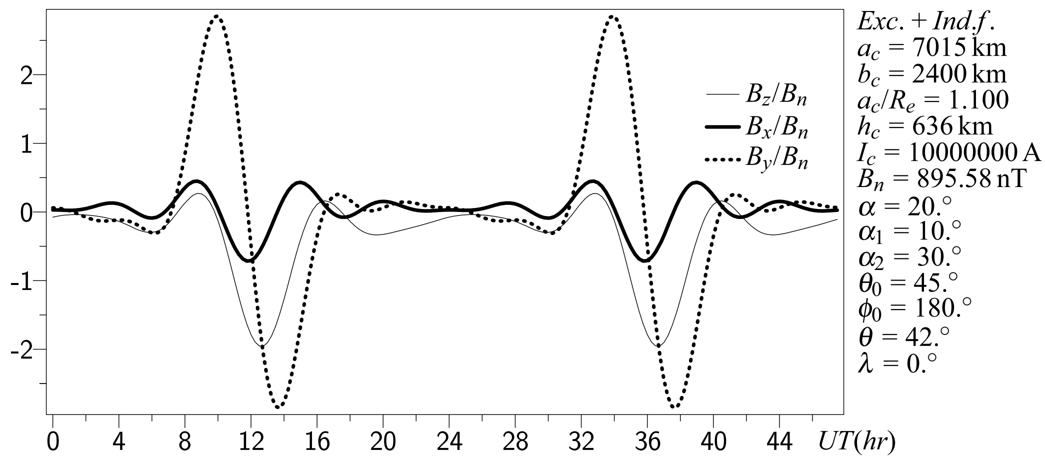


Fig. 6b. The same as in Fig. 6a, but for co-latitude θ = 42°.

Current belt exc.field

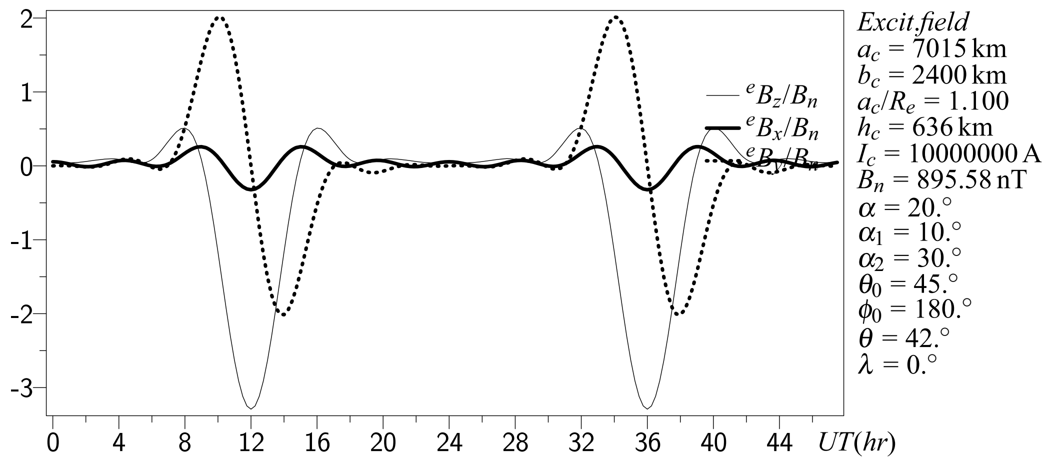


Fig. 6c. Time variations of the exciting field due to stationary Sq current belt with parameters given in Fig. 6a at the observatory θ = 42° meridian λ = 0° on the rotating Earth.

the radius of the sphere with current we put $a_c = 1.1R_e$. In Figs 6a,b we present time course of the geomagnetic components for co-latitudes $\theta = 30°, 42°$, respectively and in Fig. 6c the time variations of the exciting field on the rotating Earth for $\theta = 42°$. Comparing Figs 6b,c wee can see also the amplification of tangential components and attenuation of B_z. The time courses in Figs 6a,b are in good agreement with general features of Sq geomagnetic variations observed in polar and mid latitude geomagnetic observatories (see *Parkinson, 1983*).

Acknowledgments
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