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Dispersive conductors: The position of singularities of magnetotelluric transfer functions in the complex frequency plane

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1. Introduction

1.1 Definition of the 1D transfer function

For a 1D conductivity structure with $\hat{\mathbf{z}}$ pointing downwards, $\mathbf{E}(z,\omega) = E_x(z,\omega)\hat{\mathbf{x}}$ and a time factor $\exp(+i\omega t)$ we use Schmucker's magnetotelluric transfer function

$$C(\omega) := +\frac{E_x(0,\omega)}{i\omega B_y(0,\omega)} = -\frac{E_x(0,\omega)}{E_x'(0,\omega)}. \tag{1}$$

Its relation to impedance Z and apparent resistivity ϱ_a is

$$Z(\omega) = i\omega\mu_0 C(\omega), \quad \varrho_a(\omega) = \omega\mu_0|C(\omega)|^2. \tag{2}$$

$C(\omega)$ has the dimension of a length. In the non-dispersive case its real part is the 'center of gravity' of the induced currents.

1.2 Representation of $C(\omega)$ for a non-dispersive 1D conductivity structure

Necessary and sufficient that given data $C(\omega)$ belong to a non-dispersive 1D conductivity structure is that $C(\omega)$ can be represented as

$$C(\omega) = a_0 + \int_0^\infty \frac{a(\lambda)\,d\lambda}{\lambda + i\omega}, \quad a_0 \geq 0, \quad a(\lambda) \geq 0 \tag{3}$$

(Weidelt 1972, Parker 1980, Yee & Paulson 1988). Here

$a(\lambda)$ is a generalized function to include both the continuous and discrete part of the spectrum of decay constants. which correspond on the positive imaginary ω-axis to branch cuts and poles, respectively. $a(\lambda)$ is given by

$$a(\lambda) = -\frac{1}{\pi}\lim_{\epsilon\to o^+}\Im[C(i\lambda+\epsilon)]. \tag{4}$$

At poles, $a(\lambda) \geq 0$ means that the residua are positive. Two simple examples are

- **Uniform half-space of conductivity σ:**

$$C(\omega) = \frac{1}{\sqrt{i\omega\mu_0\sigma}} = \frac{1}{\pi\sqrt{\mu_0\sigma}}\int_0^\infty \frac{d\lambda}{\sqrt{\lambda}(\lambda+i\omega)}$$

- **Layer of conductivity σ and thickness D underlain by a perfect conductor:**

$$C(\omega) = \frac{1}{\sqrt{i\omega\mu_0\sigma}}\tanh(\sqrt{i\omega\mu_0\sigma}D) = \frac{2}{\mu_0\sigma D}\sum_{n=1}^\infty \frac{1}{\lambda_n + i\omega}, \quad \lambda_n = \frac{\pi^2(n-1/2)^2}{\mu_0\sigma D^2}.$$

In this contribution we investigate the question whether the spectral presentation is changed when a *dispersive* conductivity $\tilde{\sigma}(\omega)$ is assumed. Only if the form (3) is retained, there would exist an equivalent non-dispersive 1D model fitting dispersive data. It should be noted, however, that even in the case that the form (3) is not retained, for a *finite* set of data resulting from a dispersive model (e.g. only one frequency), an equivalent non-dispersive model may be found. However, fitting problems will arise when the data density increases.

2. A simple example of a dispersive conductivity structure

2.1 The conductivity model

We assume a simple conductivity model, consisting of a dispersive surface layer $0 \le z \le D$ with the Cole-Cole conductivity model

$$\tilde{\sigma}(\omega) = \sigma_\infty \left[1 - \frac{m}{1+(i\omega\tau)^c}\right] \tag{5}$$

underlain by a perfect conductor (see Fig. 1). For formal simplicity, the conventional time constant τ_c [$= \tau_\varrho$ in Ageev & Svetov (1999)] has been replaced by $\tau := (1-m)^{1/c}\tau_c$ [$=\tau_\sigma$ in Ageev & Svetov (1999)]. Then $C(\omega)$ is given by

$$C(\omega) = \frac{1}{k}\tanh(kD) \quad \text{with} \quad k^2 = i\omega\mu_0\tilde{\sigma}(\omega) \tag{6}$$

At present, the frequency exponent c is subject to the only condition $c \ge 0$. The polarizability m satisfies $0 \le m < 1$. – Conductivity dispersion by displacement currents will be discussed in Sect. 5.

2.2 The position of poles and zeroes of $C(\omega)$

The zeroes of $C(\omega)$ lie at frequencies where $\sinh kD = 0$, i.e. at

$$k^2D^2 = i\omega\mu_0\tilde{\sigma}(\omega)D^2 = -n^2\pi^2, \quad n = 1, 2, \ldots \tag{7}$$

and the poles occur where $\cosh kD = 0$, i.e. at

$$k^2D^2 = i\omega\mu_0\tilde{\sigma}(\omega)D^2 = -(n-1/2)^2\pi^2, \quad n = 1, 2, \ldots \tag{8}$$

As necessary conditions at zeroes and poles therefore hold

$$\Re[i\omega\tilde{\sigma}(\omega)] < 0 \quad \text{and} \quad \Im[i\omega\tilde{\sigma}(\omega)] = 0 \tag{9}$$

Before considering the general case in the next section, we will confine our attention here to two simple special cases:

a) $m = 0$**: Non-dispersive conductivity**
Zeroes and poles lie on the positive imaginary frequency axis at

$$\omega_n = \frac{i\pi^2}{\mu_0\sigma_\infty D^2} \cdot \begin{cases} n^2 \text{ for the } n\text{-th zero} \\ (n-1/2)^2 \text{ for the } n\text{-th pole} \end{cases} \tag{10}$$

For $D \to \infty$ the poles and zeroes cluster, such that a branch cut is required from $\omega = +0i$ to $\omega = \infty i$ and $C(\omega) = 1/k$.

b) $0 < m < 1$ **and** $c = 1$**: Simple Debye-dispersion**
In this case

$$\tilde{\sigma}(\omega) = \sigma_\infty \left[1 - \frac{m}{1+i\omega\tau}\right]. \tag{11}$$

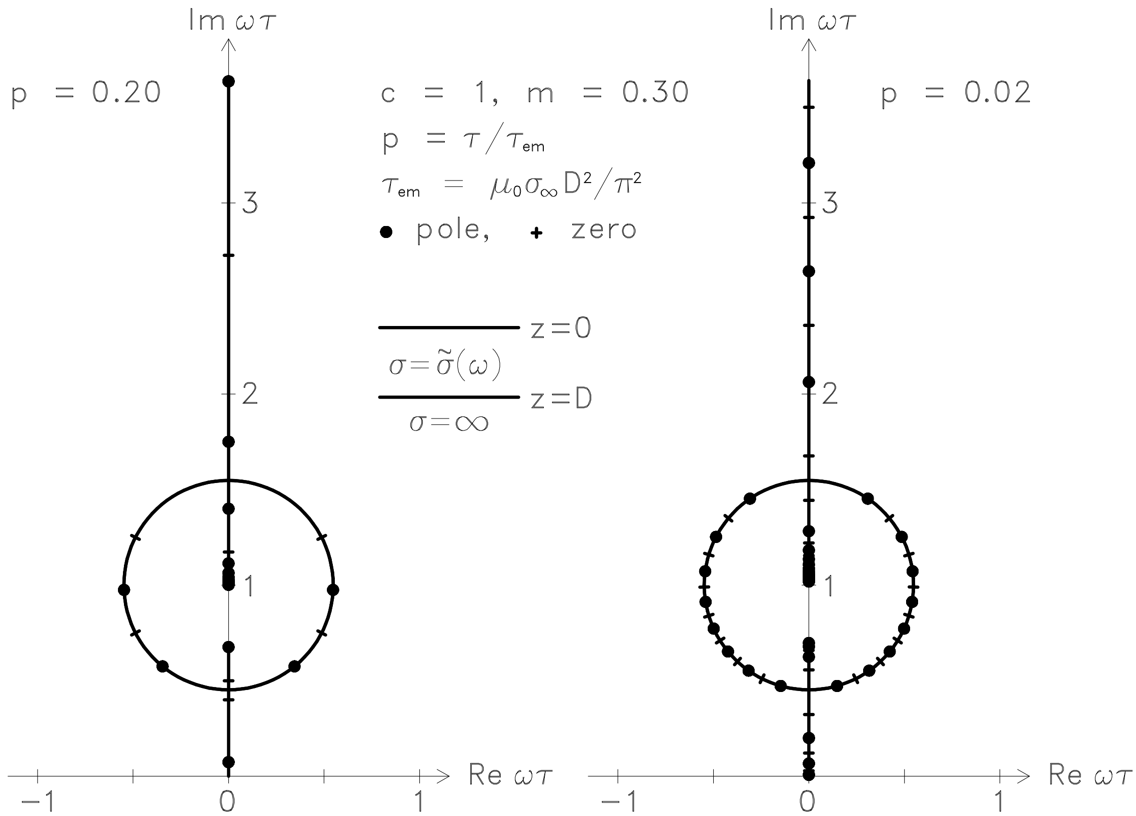

Fig. 1: Position of poles and zeroes of the magnetotelluric transfer function $C(\omega)$ for $c = 1$. The circle has its center at $\omega\tau = i$ and its radius is $\sqrt{m}$. The longest electromagnetic decay time is $\tau_{em}/4$. The parameter p represents the ratio of IP time constant τ and electromagnetic decay time. For $p \to 0$ (uniform halfspace) poles and zeroes cluster and have to be replaced by a branch cut.

Let

$$\alpha_n := \frac{\pi^2\tau}{\mu_0\sigma_\infty D^2} \cdot \begin{cases} n^2 \text{ for the } n\text{-th zero} \\ (n-1/2)^2 \text{ for the } n\text{-th pole} \end{cases} \tag{12}$$

Then the zeroes and poles lie at

$$\omega_n\tau = \begin{cases} (i/2)[\,1-m+\alpha_n \mp \sqrt{(1-m+\alpha_n)^2 - 4\alpha_n}\,], & 0 \le \alpha_n \le (1-\sqrt{m})^2 \\ (i/2)[\,1-m+\alpha_n \pm i\sqrt{4\alpha_n - (1-m+\alpha_n)^2}\,], & (1-\sqrt{m})^2 \le \alpha_n \le (1+\sqrt{m})^2 \\ (i/2)[\,1-m+\alpha_n \pm \sqrt{(1-m+\alpha_n)^2 - 4\alpha_n}\,], & \alpha_n \ge (1+\sqrt{m})^2. \end{cases} \tag{13}$$

In the dispersive case, each α_n gives rise to *two* poles or zeroes. The singular curves consist of the positive imaginary axes-sections $0 \le \Im\omega\tau \le 1-m$ and $1 \le \Im\omega\tau < \infty$ and the circle $|\omega\tau - i| = \sqrt{m}$, see Fig. 1. In the first line of (13) the minus refers to poles and zeroes on the section $0 \le \Im\omega\tau \le 1-\sqrt{m}$ outside the circle and the plus to the section $1-\sqrt{m} \le \Im\omega\tau \le 1-m$ inside the circle. Similarly, in the third line the plus gives poles and zeroes on the section $1+\sqrt{m} \le \Im\omega\tau < \infty$ outside the circle and the minus poles and zeroes on the section $1 \le \Im\omega\tau \le 1+\sqrt{m}$ inside the circle with a clustering at its center $\omega\tau = i$. In the limit $m \to 0$ we have for poles and zeroes *outside* the circle $\omega_n\tau \to i\alpha_n$, in agreement with (10). All poles and zeroes *inside* the circle tend to the $\omega\tau = i$. In a Mittag-Leffler reconstruction of $C(\omega)$ from the poles they are entering with zero weight (vanishing residuum).

The poles and zeroes of Eq. (13), displayed in Fig. 1, show that in general poles and zeroes outside the positive imaginary axis occur, which precludes the existence of an equivalent non-dispersive conductivity model. In the case that $\alpha_1 > (1+\sqrt{m})^2$ or

$$\frac{\pi^2\tau}{\mu_0\sigma_\infty D^2} > 4(1+\sqrt{m})^2,$$

it follows from (13) that all poles of $C(\omega)$ are positioned on the positive imaginary axis, as in the non-dispersive case. However, the residua belonging to the poles inside the circle on the line $1 \le \Im\omega\tau \le 1+\sqrt{m}$ turn out to be *negative*. Therefore also in this case no non-dispersive model exists.

3. Poles and zeroes in the general case

The singular lines satisfying the necessary conditions (9) are shown in Fig. 2 for several values of the frequency exponent c. In addition branch cuts exist for $c \neq 1$. The spread between the left and right line for $c < 1$ infinitely increases in the limit $\Im\omega\tau \to \infty$. (This also holds for $c = 0.95$.) For the (perhaps unphysical) values $c > 1$ the circle at $c = 1$ disintegrates into two arcs with two well-defined endpoints at the points, where $\tilde{\sigma}(\omega)$ either grows to infinity or vanishes, i.e where $1+(i\omega\tau)^c = 0$ or $1+(i\omega\tau)^c = m$. Poles and zeroes cluster at the former point (marked by a heavy dot). It is easily found that both endpoints have the azimuth $\varphi = \pm\pi/c - \pi/2$, with the azimuth φ counted from the positive real frequency axis. With increasing c the singular lines migrate towards the real frequency axis, such that $C(\omega)$ becomes increasingly singular there. The limiting value $c_0(m)$ of c with its singular line just touching the real frequency axis is determined by the condition that according to (9) the equation $\Im[i\omega\tilde{\sigma}(\omega)] = 0$ or equivalently $\Re[\tilde{\sigma}(\omega)] = 0$ has exactly one real (positive) solution. This gives

$$c_0(m) = 1 + (4/\pi)\tan^{-1}\sqrt{1-m}. \qquad (14)$$

Examples: $c_0(0.1) = 1.97$, $c_0(0.3) = 1.89$, $c_0(0.5) = 1.78$, $c_0(0.7) = 1.64$, $c_0(0.9) = 1.39$. For $c \ge c_0(m)$ the singularities are no longer confined to the upper frequency half-plane and therefore we are no longer facing a *causal* system.

For any choice of $p = \tau/\tau_{em}$ the poles and zeroes of $C(\omega)$ are located on the full lines. The actual position is obtained after specifying p. An example is shown in Fig. 3. Displayed is the sign of the spectral function defined according to (4). For $c < 1$ it is negative and for $c > 1$ it is positive. In the case of complex poles also the residua are complex. Therefore only the real part is shown by circles, which have an area proportional to the real part. The essentially positive nature of $C(\omega)$ is carried for $c < 1$ by the poles and for $c > 1$ by the branch cut. The affinity between the 'circle' for $c = 1$ and the 'arcs' for $c = 1.05$ is illustrated by the negative signs of the residua at the ends of the arc, which result from the negative residua inside the circle for $c = 1$. – The greatest frequency exponent $c = 1.78$ is associated with two arcs touching the real frequency axis.

For $p \to 0$ the sequence the of poles (and zeroes) forms a dense line (branch cut). In this case only the endpoints (branch points) are fixed, which means that – for instance – the endpoints of the arcs can be connected by a straight line. This case is considered in more detail in the next section.

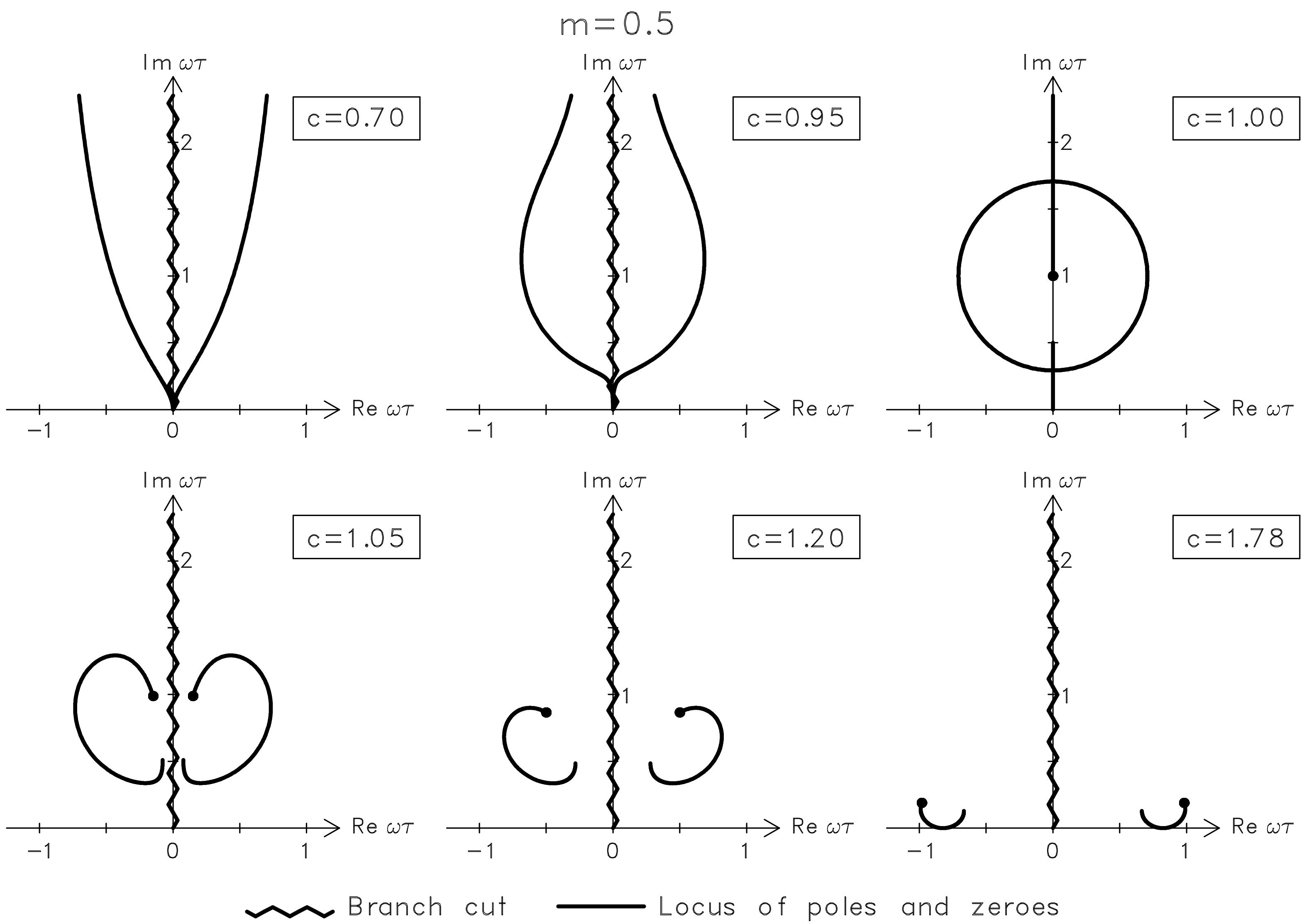

Fig. 2: Poles and zeroes of the magnetotelluric transfer function $C(\omega)$ are positioned on the full lines. Except for $c = 1$ there is in addition a branch cut from $\omega = +i0$ to $\omega = +i\infty$. For $c > 1$ the circle occurring at $c = 1$ disintegrates into two arcs. The heavy dots, corresponding to $\tilde{\sigma}(\omega) = \infty$, mark cluster points of poles.

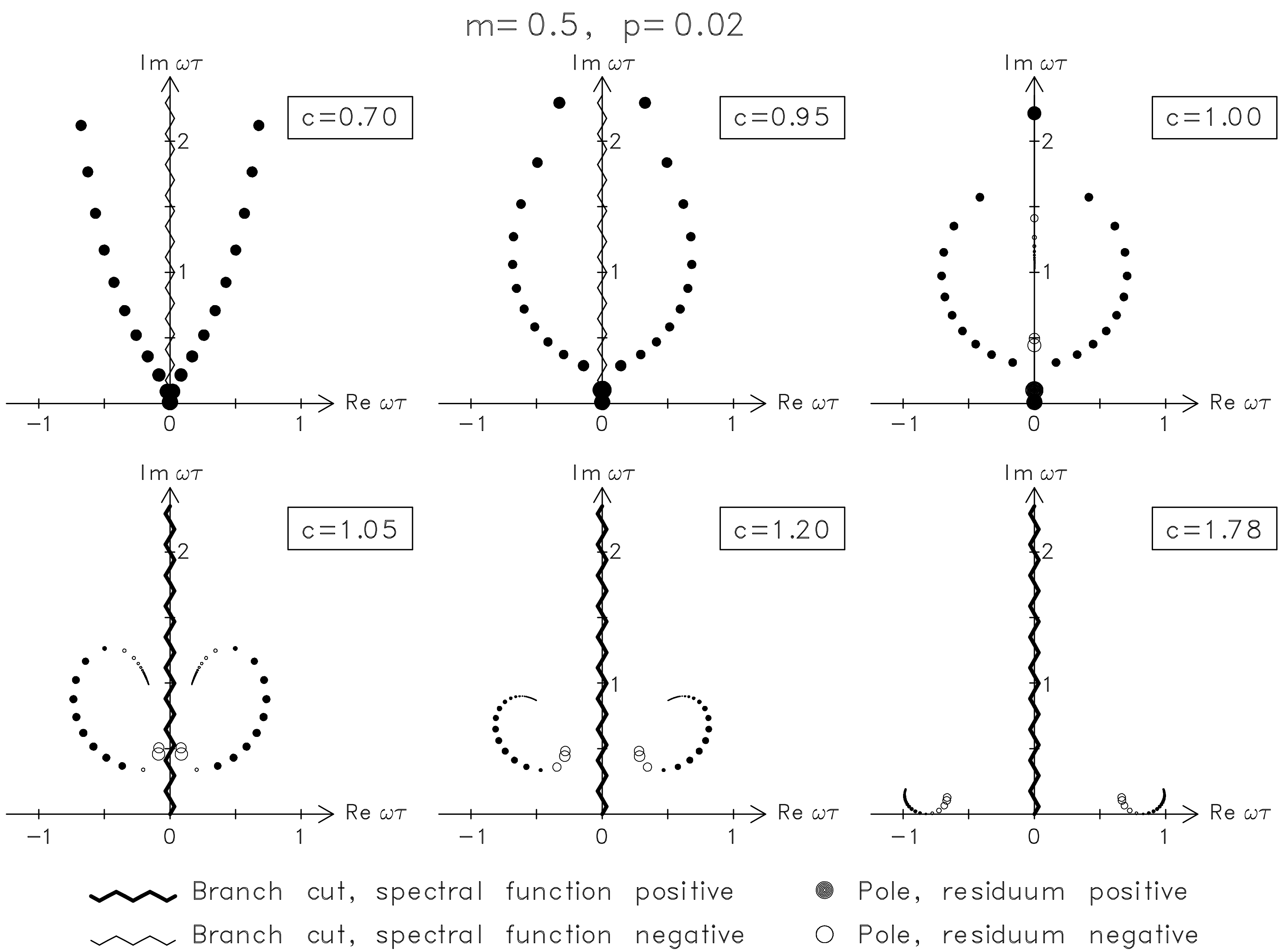

Fig. 3: Position of poles and branch cuts of the magnetotelluric transfer function $C(\omega)$ for various frequency exponents c and a fixed value $p = \tau/\tau_{em} = 0.02$ (see also Fig. 1). Given are also the signs both of the residua and the spectral function at the branch cut. Note that the sign of the spectral functions changes at $c = 1$. – The area of the dots matking the position of the residua is in proportion to the real part of the residuum.

4. Dispersive uniform half-space

Special attention requires the case $D \to \infty$, the limiting model of a uniform dispersive half-space. As already mentioned, in the corresponding limit $p \to 0$ the density of poles (and zeroes) increases – see Figs. 1 and 3 – and ultimately they form a continuous line = branch cut; branch cuts, however, can be deformed in the complex plane. For $0 \le c \le 1$ the branch cuts off the positive imaginary axis, originating at $\omega = 0$, can unified with the already existing branch cut along this line. For $1 < c \le 2$ Fig. 3 shows two circular arc off the positive imaginary axis with the endpoints

$$\omega\tau = (1-m)^{1/c} \exp\{i\pi(\pm 1/c - 1/2)\} \quad \text{and} \quad \omega\tau = \exp\{i\pi(\pm 1/c - 1/2)\}$$

respectively (+ and − refer to $\Re\omega > 0$ and $\Re\omega < 0$). These endpoints can be connected by a straight branch cuts, such that also in the range $c_0 < c \le 2$ all singularities and zeroes are situated in the upper ω-plane (whereas for $D < \infty$ the restriction $c \le c_0$ holds). The absence of singularities and zeroes in the lower frequency plane is the necessary condition for the existence of dispersion relations connecting apparent resistivity $\varrho_a(\omega)$ and phase $\varphi(\omega) = \arg(Z(\omega))$, see (2). The existence of branch cuts away from the positive imaginary axis, however, prohibits a non-dispersive 1D interpretation of the dispersive response for $1 < c \le 2$.

The response function is

$$C(\omega) = \frac{1}{\sqrt{i\omega\mu_0\tilde{\sigma}(\omega)}}$$

with $\tilde{\sigma}(\omega)$ given in (5). For $0 \le c \le 1$ it admits the spectral presentation (3), where $a_0 = 0$ and $a(\lambda)$ given according to (4) by

$$a(\lambda) = \frac{1}{\pi\sqrt{\lambda\mu_0\sigma_\infty}} \cdot \Re\left\{1 - \frac{m}{1 + (\lambda\tau)^c \exp(i\pi c)}\right\}^{-1/2},$$

A simple analysis reveals that $a(\lambda) \ge 0$. Therefore there exists an equivalent non-dispersive 1D model completely explaining the dispersive data. In particular, for the simple case $c = 1$ we obtain

$$a(\lambda) = \frac{1}{\pi\sqrt{\lambda\mu_0\sigma_\infty}} \cdot \begin{cases} \sqrt{(1-\lambda\tau)/(1-m-\lambda\tau)}, & 0 \le \lambda\tau < 1-m, \\ 0, & 1-m < \lambda\tau < 1, \\ \sqrt{(\lambda\tau - 1)/(\lambda\tau + m - 1)}, & \lambda\tau > 1 \end{cases}$$

No closed form of the associated non-dispersive conductivity distribution was found, but the first terms of a series presentation in the non-dimensional variable $x := \mu_0\sigma_\infty z^2/\tau$ can be given:

$$\frac{\sigma(z)}{\sigma_\infty} = 1 - 2mx + \frac{m(9m+2)x^2}{3} - \frac{4m(45m^2+27m+1)x^3}{45} + \frac{m(1575m^3+1728m^2+243m+2)x^4}{315} - \cdots$$

Fig. 4 shows apparent resistivity and phase for four values of c. The first two admit even a non-dispersive 1D interpretation. For $c > c_0$ (not displayed) the phase can attain negative values. This becomes obvious in the simple limiting case $c = 2$:

$$\varphi(\omega) = \pi/2 + \arg C(\omega) = \begin{cases} +\pi/4, & 0 \le \omega\tau < \sqrt{1-m}, \\ -\pi/4, & \sqrt{1-m} \le \omega\tau < 1, \\ +\pi/4, & \omega\tau > 1, \end{cases}$$

$$\varrho_a(\omega) = \omega\mu_0|C(\omega)|^2 = \frac{1}{\sigma_\infty} \cdot \left|\frac{1-\omega^2\tau^2}{1-m-\omega^2\tau^2}\right|.$$

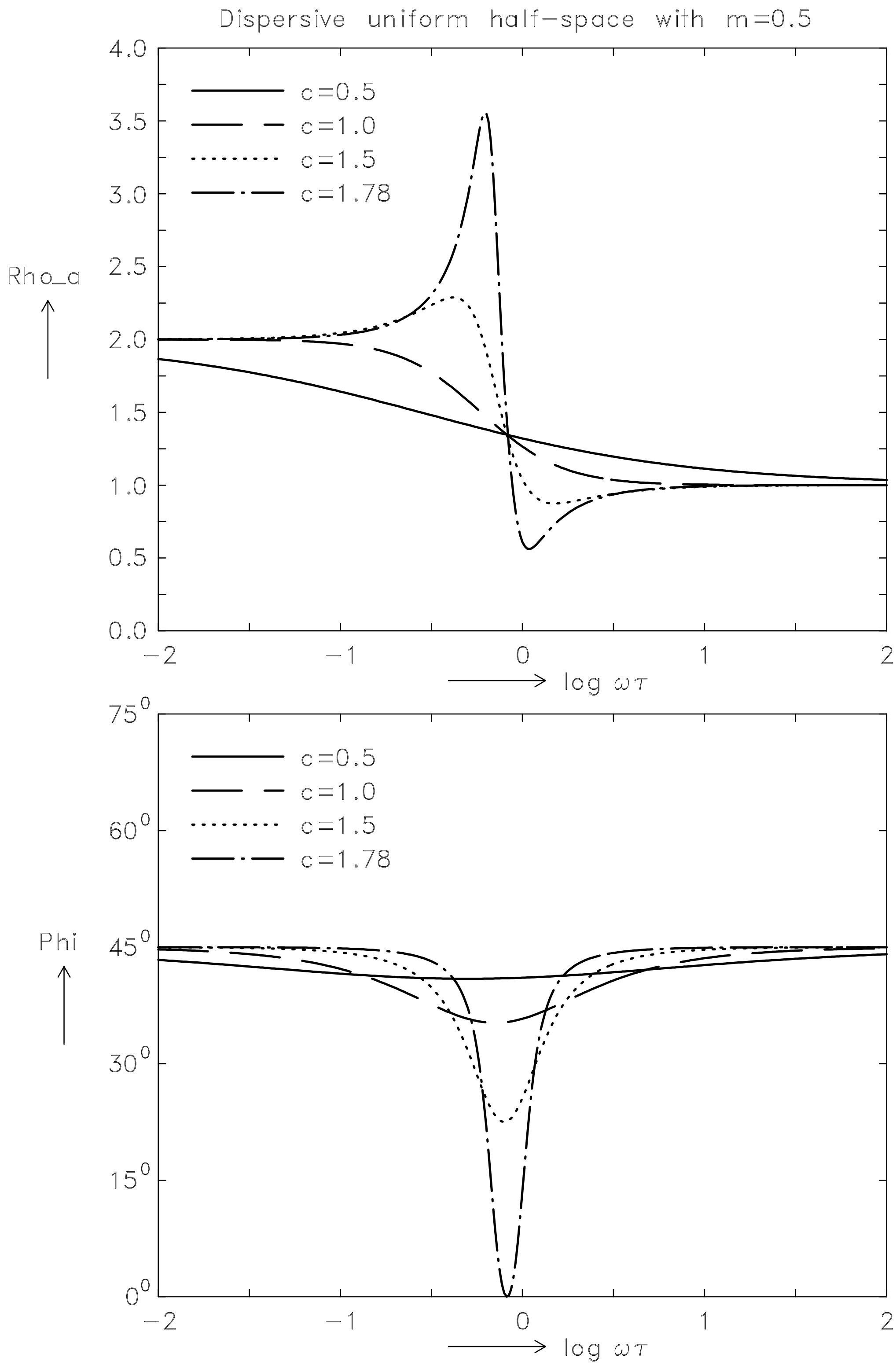

Fig. 4: Apparent resistivity and phase of a dispersive uniform half-space. The apparent resistivity is normalized with respect to $\varrho_\infty = 1/\sigma_\infty$. In the range $0 \leq c \leq 1$ a non-dispersive 1D interpretation of the data is possible, because $C(\omega)$ can be presented by branch cuts along the positive imaginary frequency axis with a non-negative spectral function. For $1 < c \leq 2$ there are in addition two branch cuts off the positive imaginary ω-axis. Therefore no non-dispersive 1D interpretation is possible. For $c_0 = 1.78 < c \leq 2$ [see Eq. (14)] the phase can become negative. Dispersion relations connecting phase and apparent resistivity exist in the full range $0 \leq c \leq 2$.

5. Brief look on frequency dispersion by displacement currents

In this last section we are throwing a look on the analytic properties of $C(\omega)$ if displacement currents are taken into account. The uniform 'dispersive' conductivity

$$\tilde{\sigma}(\omega) = \sigma + i\omega\epsilon \tag{15}$$

with the permittivity ϵ is assumed in $0 \le z < D$ and $\tilde{\sigma}(\omega) = \infty$ in $z > D$. The response is again (6) with $\tilde{\sigma}(\omega)$ given in (15).

With (7)-(9) we find as locations of poles and zeroes of $C(\omega)$

- **Position on the positive imaginary axis:**

$$\omega_n^{\pm} = \frac{i\sigma \pm i\sqrt{\sigma^2 - \gamma_n^2}}{2\epsilon}, \quad \gamma_n \le \sigma,$$

- **Position away from the positive imaginary axis:**

$$\omega_n^{\pm} = \frac{i\sigma \pm \sqrt{\gamma_n^2 - \sigma^2}}{2\epsilon}, \quad \gamma_n \ge \sigma,$$

with $\gamma_n := (2n-1)\pi\sqrt{(\epsilon/\mu_0)}/D$ for the n-th pole and $\gamma_n := 2n\pi\sqrt{(\epsilon/\mu_0)}/D$ for the n-th zero, $n = 1, 2, 3, \ldots$

There are two branches of poles and zeroes (identified by the superscripts + and $-$): For small n, i.e. $\gamma_n < \sigma$, the branch ω_n^- starts at the origin and moves the positive imaginary axis upwards and ω_n^+ moves from $i\sigma/\epsilon$ downwards. They meet for $\gamma_n = \sigma$ at $i\sigma/(2\epsilon)$ and then spread out horizontally to $i\sigma/(2\epsilon) \pm \infty$. The quasistatic case (10) evolves from ω_n^- with $\gamma_n < \sigma$:

$$\omega_n^- = \frac{i\gamma_n^2}{2\epsilon(\sigma + \sqrt{\sigma^2 - \gamma_n^2})} \to \frac{i\pi^2}{\mu_0\sigma D^2} \cdot \begin{cases} n^2 \text{ for the } n\text{-th zero} \\ (n-1/2)^2 \text{ for the } n\text{-th pole} \end{cases}$$

Since all poles and zeroes of $C(\omega)$ are lying in $\Im\omega > 0$, dispersion relations between ϱ_a and φ exist.

For $D < \infty$ the existence of poles away from the positive imaginaries prohibits a non-dispersive 1D interpretation. The situation changes for $D \to \infty$, when the conductor degenerates into a uniform half-space. Since in this limit $\gamma_n \to 0$, no poles off the positive imaginary axis occur. The response becomes

$$C(\omega) = \frac{1}{\sqrt{i\omega\mu_0(\sigma + i\omega\epsilon)}} = \frac{1}{\pi\sqrt{\epsilon\mu_0}} \int_0^b \frac{d\lambda}{\sqrt{\lambda(b-\lambda)}(\lambda + i\omega)}, \quad b := \sigma/\epsilon.$$

Since the spectral function $a(\lambda)$ is positive, see Eq. (3), a non-dispersive 1D interpretation exists. In this case it is simply an almost uniformly laminated conductor consisting of an infinite stack of thin sheets. Each sheet is described by its conductance (= integrated conductivity) S_n and its depth z_n, $n = 0, 1, 2, \ldots$ It results

$$z_n = (2n/\sigma)\sqrt{\epsilon/\mu_0}, \ n \ge 0 \ \text{ and } \ S_0 = \sqrt{\epsilon/\mu_0}, \ S_n = 2S_0, n \ge 1.$$

Here, $(2/\sigma)\sqrt{\epsilon/\mu_0}$ is the high frequency limit of the electromagnetic penetration depth and $1/S_0$ is the plane-wave impedance (= 377 Ω for $\epsilon = \epsilon_0$).

For the displacement current we have encountered the same situation as for Cole-Cole dispersion: In both cases 1D interpretation becomes possible only for the uniform half-space, but not even for the simple layered model considered as example.

6. Conclusions

We have considered the position of singularities of the magnetotelluric response function $C(\omega)$ for a simple model consisting of a uniform layer of thickness D over a perfect conductor, assuming in the layer Cole-Cole dispersion or simple dispersion by displacement currents. For Cole-Cole dispersion we have studied frequency exponents c in the range $0 \leq c \leq 2$, although in practice c is assumed to be significantly less than unity. This extended range is at least physically possible, only for $c > 2$ causality breaks down.

The results are summarized as follows:

- For $D < \infty$ no dense and exact data set resulting from the dispersive conductor can be interpreted in terms of a non-dispersive conductivity distribution. However, for $D = \infty$ (uniform half-space) a non-dispersive 1D interpretation becomes possible (in the Cole-Cole case for frequency exponents $0 \leq c \leq 1$).
- Dispersion relations between apparent resistivity $\varrho_a(\omega)$ and the phase of $C(\omega)$ exist also for dispersive data with the frequency exponent restricted for Cole-Cole data in the range $c < c_0(m)$, where $c_0(m)$ is given in (14). In the case $D = \infty$, dispersion relations exist even for $0 \leq c \leq 2$.

The results have been obtained for a special simple conductivity profile, but an extension to general 1D profiles with a dispersive section is possible.

The bad news is that so far nobody has reliably identified dispersion effects in magnetotelluric data. Therefore at present this study reduces unfortunately to a mere exercise in the theory of complex variables.

7. References

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